Higher-Level Paradoxes and Substructural Solutions*

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Abstract

There have been recent arguments against the idea that substructural solutions are uniform. The claim is that even if the substructuralist solves the common semantic paradoxes uniformly by targeting Cut or Contraction, with additional machinery, we can construct higher-level paradoxes (e.g., a higher-level Liar, a higher-level Curry, and a meta-validity Curry). These higher-level paradoxes do not use metainferential Cut or Contraction, but rather, higher-level Cuts and higher-level Contractions. These kinds of paradoxes suggest that targeting Cut or Contraction is not enough for solving semantic paradoxes; the substructuralist must target Cut of every level or Contraction of every level to solve the paradoxes. Hence, the substructuralists do not provide as uniform of a solution as they hoped they did. In response, we argue that the substructuralists need not admit these additional machineries. In fact, they are redundant in light of the validity predicate (i.e., there is no gain in terms of expressive power). The validity predicate is powerful enough to creep these paradoxes in the object level. The substructuralist does not need to ascend to metainferences to construct higher-level paradoxes. Moreover, there is a reading available to the substructuralist such that all the higher-level structural rules would collapse to instances of the object-level structural rules (e.g., meta$_n$Cut and meta$_n$Contraction would become instances of Cut and Contraction). We then address Barrio et al.'s worry that the validity predicate has its shortcomings; the substructuralist cannot internalize some of its metarules. We claim that the validity of metarules can be internalized without the need to strengthen the validity predicate. However, a problem raised by Barrio et al. is still present—the problem of internalizing unwanted instances of Cut in Cut-free approaches. We argue that this internalization problem is not unique to the validity predicate; the same problem is present with other problematic predicates, such as the truth predicate and the provability predicate.

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There has been a charge against the substructuralists that their solutions are not as uniform as they took it to be. The idea is that we can construct higher-level paradoxes where Cut, Contraction, and Identity play no role, but rather, higher versions of these structural rules are employed. Hence, rejecting or restricting a structural rule is not enough to avoid all the semantic paradoxes.

We start the paper by explicating this charge (§1). We then make the case that there is no reason to ascend the suggested hierarchy, since all of these higher-level paradoxes are expressible in the object language using the validity predicate (§2). After that, we address the worry that the validity predicate is not strong enough to internalize the metarules of our logic (§3). We explore two possible options to internalize the metarules without the need to strengthen the validity predicate. Finally, we address a pressing issue that shows that the validity predicate internalizes metainferences that are not admissible in the system. We claim that this problem is not unique to the validity predicate; it is present in other predicates prone to paradoxicality.

1 The Charge

A sequent (an inference) is a relationship between sequences of formulas. For instance, in the sequent $\Gamma \vdash \Delta$, we say the disjunction of the formulas in Δ follow from the conjunction of the formulas in Γ. A metainference, on the other hand, is a relationship between sequents (inferences). For example:

$$\frac{\Gamma \vdash \Delta}{\Sigma \vdash \Xi}$$

We say that the sequent $\Sigma \vdash \Xi$ follows from the sequent $\Gamma \vdash \Delta$. The metainference itself can be seen as a sequent of sequents. Thus, we can represent the horizontal line as a turnstile, but since it is a sequent of sequents rather than formulas, it is of a higher level than a sequent of formulas. Hence, we will index the turnstiles. We will use $\vdash_0$ for sequents of formulas (i.e., object level) and $\vdash_1$ for sequents of sequents of formulas. We can represent the previous metainference as follows: $\{\Gamma \vdash_0 \Delta\} \vdash_1 \{\Sigma \vdash_0 \Xi\}$.

In ([18], manuscript version), Priest borrows Brian Porter's observations regarding higher-level paradoxes [17] in order to argue against substructural solutions. The idea is that even if the substructuralist solves the paradoxes uniformly by targeting Cut or Contraction, with "additional machinery", we can construct higher-level paradoxes. The higher-level paradoxes do not use metainferential Cut or metainferential Contraction, but rather, higher-level Cuts and higher-level Contractions. For instance, the following are the rules Cut_2 and $\mathrm{Contraction}_2$:

$$\frac{\Gamma \vdash_1 \eta, \Delta \qquad \Gamma', \eta \vdash_1 \Delta'}{\Gamma, \Gamma' \vdash_1 \Delta, \Delta'}\,\mathrm{Cut}_2$$

$$\frac{\Gamma, \eta, \eta \vdash_1 \Delta}{\Gamma, \eta \vdash_1 \Delta}\ \text{Contraction}_2 \qquad \frac{\Gamma \vdash_1 \eta, \eta, \Delta}{\Gamma \vdash_1 \eta, \Delta}\ \text{Contraction}_2$$

where Γ, Γ', Δ, and Δ' are finite multisets of sequents of level-0, and η is a sequent of level-0.

Among the additional machineries that Priest adds, are names for sequents of lower level and rules for the truth predicate to apply to sequents ([18], p.17 of manuscript version):

$$\frac{\eta \vdash_1 \xi}{(\top \vdash_0 Tr(\ulcorner \eta \urcorner)) \vdash_1 \xi}\ \text{Tr}_2 \qquad \frac{\eta \vdash_1 \xi}{\eta \vdash_1 (\top \vdash_0 Tr(\ulcorner \xi \urcorner))}\ \text{Tr}_2$$

where "$\top$" indicates an empty antecedent in the sequent, "$\ulcorner\urcorner$" indicates a naming device that names object-level sequents (level-0), and η and ξ are sequents of level-0. Moreover, for every inference rule in our object language, we have higher-level versions of them. For example, we have $(\vdash_1 \neg)$ where negation is applied to a sequent of level-0:

$$\frac{\Gamma, \eta \vdash_1 \Delta}{\Gamma \vdash_1 \neg\eta, \Delta}\ \vdash_1 \neg$$

From there, we can construct a higher-level Liar and a higher-level Curry. For the higher-level Liar, let λ be $\neg(\top \vdash_0 Tr(\ulcorner \lambda \urcorner))$. So λ says something along the lines 'it is not a theorem that I am true':

$$\dfrac{\dfrac{\dfrac{\dfrac{\lambda \vdash_1 \lambda}{(\top \vdash_0 Tr(\ulcorner \lambda \urcorner)) \vdash_1 \lambda}\ \text{Tr}_2}{\vdash_1 \neg(\top \vdash_0 Tr(\ulcorner \lambda \urcorner)), \lambda}\ \vdash_1 \neg}{\vdash_1 \lambda, \lambda}\ \text{Def of } \lambda}{\vdash_1 \lambda}\ \text{Contraction}_2$$

Call this derivation D_0. Similarly for the other side:

$$\dfrac{\dfrac{\dfrac{\dfrac{\lambda \vdash_1 \lambda}{\lambda \vdash_1 (\top \vdash_0 Tr(\ulcorner \lambda \urcorner))}\ \text{Tr}_2}{\lambda, \neg(\top \vdash_0 Tr(\ulcorner \lambda \urcorner)) \vdash_1}\ \neg \vdash_1}{\lambda, \lambda \vdash_1}\ \text{Def of } \lambda}{\lambda \vdash_1}\ \text{Contraction}_2$$

Call this derivation D_1. Finally:

$$\dfrac{\dfrac{[D_0]}{\vdash_1 \lambda} \qquad \dfrac{[D_1]}{\lambda \vdash_1}}{\vdash_1}\ \text{Cut}_2$$

For the higher-level Curry, let γ be $(\top \vdash_0 Tr(\ulcorner\gamma\urcorner)) \to \varphi$ where φ is an arbitrary sequent of level-0:

$$\dfrac{\dfrac{\dfrac{\dfrac{\gamma \vdash_1 \gamma}{\gamma \vdash_1 (\top \vdash_0 Tr(\ulcorner\gamma\urcorner))}\,\mathrm{Tr}_2 \qquad \varphi \vdash_1 \varphi}{\gamma, (\top \vdash_0 Tr(\ulcorner\gamma\urcorner)) \to \varphi \vdash_1 \varphi}\,{\to}{\vdash_1}}{\gamma, \gamma \vdash_1 \varphi}\,\text{Def of }\gamma}{\gamma \vdash_1 \varphi}\,\text{Contraction}_2$$

Call this derivation D_0.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[D_0]}{\gamma \vdash_1 \varphi}}{(\top \vdash_0 Tr(\ulcorner\gamma\urcorner)) \vdash_1 \varphi}\,\mathrm{Tr}_2}{\vdash_1 (\top \vdash_0 Tr(\ulcorner\gamma\urcorner)) \to \varphi}\,{\vdash_1}{\to}}{\vdash_1 \gamma}\,\text{Def of }\gamma \qquad \dfrac{[D_0]}{\gamma \vdash_1 \varphi}}{\vdash_1 \varphi}\,\mathrm{Cut}_2$$

Thus, we proved φ out of thin air without having to use a metainferential Cut or a metainferential Contraction.

Similarly, in [17], Brian Porter presented metainferential versions of the validity Curry paradox. Porter builds an infinite hierarchy of metainferences, but for the sake of brevity, we will stick to one level higher than the validity Curry. Moreover, Porter uses indices for different levels of the validity predicate, however, we will omit these indices for readability and because the same argument can be carried out without the need of these indices.

In order to carry out his argument, Porter presents two metametainference rules for the metainferential validity Curry analogous to Beall and Murzi's VP and VD in [5]. The first rule, VP_2 goes as follows:[1]

$$\dfrac{\{\Gamma \vdash_0 \Delta\} \vdash_1 \{\Sigma \vdash_0 \Pi\}}{\vdash_1 \{\vdash_0 Val(\ulcorner\Gamma \vdash_0 \Delta\urcorner, \ulcorner\Sigma \vdash_0 \Pi\urcorner)\}}\,\mathrm{VP}_2$$

In other words, if

[1] Porter uses $\vdash_1$ for object-level inferences and $\vdash_2$ for the next level. We will keep them $\vdash_0$ and $\vdash_1$ respectively as we have done above.

$$\frac{\Gamma \vdash_0 \Delta}{\Sigma \vdash_0 \Pi}$$

is valid, then

$$\frac{}{\vdash_0 Val(\ulcorner \Gamma, \Delta \urcorner, \ulcorner \Sigma, \Pi \urcorner)}$$

is valid ([17], p. 91-92).

The second rule, VD_2, is stated as follows:

$$\frac{}{\{\Gamma \vdash_0 \Delta, \vdash_0 Val(\ulcorner \Gamma \vdash_0 \Delta \urcorner, \ulcorner \Sigma \vdash_0 \Pi \urcorner)\} \vdash_1 \{\Sigma \vdash_0 \Pi\}}\ \text{VD}_2$$

In other words,

$$\frac{\Gamma \vdash_0 \Delta, \quad \vdash_0 Val(\ulcorner \Gamma, \Delta \urcorner, \ulcorner \Sigma, \Pi \urcorner)}{\Sigma \vdash_0 \Pi}$$

With VP_2, VD_2, Cut_2, and $Contraction_2$ in hand, we can construct Porter's Meta-validity Curry.[2] Let π be $Val(\ulcorner \vdash_0 \pi \urcorner, \ulcorner \vdash_0 \bot \urcorner)$ where $\bot$ stands for something absurd:[3]

$$\dfrac{\dfrac{\dfrac{}{\{\vdash_0 \pi, \vdash_0 Val(\vdash_0 \pi, \vdash_0 \bot)\} \vdash_1 \{\vdash_0 \bot\}}\ \text{VD}_2}{\{\vdash_0 \pi, \vdash_0 \pi\} \vdash_1 \{\vdash_0 \bot\}}\ \text{def of } \pi}{\{\vdash_0 \pi\} \vdash_1 \{\vdash_0 \bot\}}\ \text{Contraction}_2$$

Call this derivation D_0.

$$\dfrac{\dfrac{\dfrac{\dfrac{[D_0]}{\{\vdash_0 \pi\} \vdash_1 \{\vdash_0 \bot\}}}{\vdash_1 \{\vdash_0 Val(\vdash_0 \pi, \vdash_0 \bot)\}}\ \text{VP}_2}{\vdash_1 \{\vdash_0 \pi\}}\ \text{def of } \pi \qquad \dfrac{[D_0]}{\{\vdash_0 \pi\} \vdash_1 \{\vdash_0 \bot\}}}{\vdash_1 \{\vdash_0 \bot\}}\ \text{Cut}_2$$

[2] A similar though slightly different version of a metavalidity Curry can be found in Rohan French's ([8], p.126).

[3] Ignoring the name forming device "$\ulcorner\urcorner$" to simplify the proof.

Thus, $\vdash_1 \vdash_0 \perp$ is a valid inference, which renders the logic to be trivial since we can choose anything in $\perp$'s place ([17], p. 93-94).

As a result, Porter claims that the substructuralist cannot provide a uniform solution that extends to the metainferential validity Curry. In order to avoid the metainferential validity Curry, we must give up either VP_2, VD_2, Cut_2, or $\mathrm{Contraction}_2$—Cut and Contraction play no role in the metainferential validity Curry.

Thus, these kinds of paradoxes suggest that targeting Cut or Contraction is not enough for solving semantic paradoxes; the substructuralist must target Cut of every level[4] or Contraction of every level to solve the paradoxes uniformly ([17], [18]). Hence, the substructuralists do not provide as uniform of a solution as they hoped they did.

2 Collapsing the Hierarchy

The substructuralist need not admit these additional machineries. In fact, they are redundant in light of the validity predicate. It is not clear that there is any gain in terms of expressive power. After all, as Priest has mentioned, these sequents are "simply metatheoretic conditionals" ([18], p.17 in fn. of manuscript version). But that is what the validity predicate is. All the aforementioned paradoxes can be expressed using our regular validity predicate, and thus, we do not need to ascend to further levels. Following our recipe for paradox in [1], the sentences above would be flagged as paradoxical if we translate them with a validity predicate. After all, they are sentences that are equivalent to their own partial negative modalities (and these partial negative modalities contain partially transparent predicates). m is said to be a *partial negative modality* if $\vdash mA$ is validly deducible from $A \vdash$ ([1], p.267). A *partially transparent predicate* is a predicate that obeys the following two rules ([1], p.267-268):

(a) If $\vdash \varphi$ then $\vdash P(\ulcorner \varphi \urcorner)$

(b) $P(\ulcorner \varphi \urcorner) \vdash \varphi$

Hence, the validity predicate is partially transparent ([1], p.270).

We can translate λ above, namely $\neg(\top \vdash_0 Tr(\ulcorner \lambda \urcorner))$, into the following sentence $\neg Val(\top, Tr(\ulcorner \lambda \urcorner))$. It turns out that $\neg Val(\top, Tr(\ulcorner\urcorner))$ is a partial negative modality:[5] [6]

[4] For systems that target Cut at every level by constructing an ST-hierarchy, see [3, 4, 15].

[5] We are still ignoring the name forming device "$\ulcorner\urcorner$" in our validity predicate for readability purposes. We will keep omitting the name forming device throughout most of the paper.

[6] Following our previous steps in [1], we are helping ourselves with the following equivalences of VD here and elsewhere where Cut is available (i.e., when flagging the paradoxes via

$$
\begin{prooftree}
\AxiomC{$\vdash \top$}
\AxiomC{$\lambda \vdash$}
\RightLabel{\scriptsize Transparency}
\UnaryInfC{$Tr(\ulcorner \lambda \urcorner) \vdash$}
\RightLabel{\scriptsize $\rightarrow\vdash$}
\BinaryInfC{$\top \rightarrow Tr(\ulcorner \lambda \urcorner) \vdash$}
\RightLabel{\scriptsize VD}
\UnaryInfC{$Val(\top, Tr(\ulcorner \lambda \urcorner)) \vdash$}
\RightLabel{\scriptsize $\vdash \neg$}
\UnaryInfC{$\vdash \neg Val(\top, Tr(\ulcorner \lambda \urcorner))$}
\end{prooftree}
$$

In fact, the truth predicate here is not necessary, but we will keep it to match Priest's λ. Now, letting λ be its own partial negative modality, $\neg Val(\top, Tr(\ulcorner \lambda \urcorner))$, we can reconstruct the higher-level liar in the object level:

$$
\begin{prooftree}
\AxiomC{$\vdash \top$}
\AxiomC{$\lambda \vdash \lambda$}
\RightLabel{\scriptsize Transparency}
\UnaryInfC{$Tr(\ulcorner \lambda \urcorner) \vdash \lambda$}
\RightLabel{\scriptsize $\rightarrow\vdash$}
\BinaryInfC{$\top \rightarrow Tr(\ulcorner \lambda \urcorner) \vdash \lambda$}
\RightLabel{\scriptsize VD}
\UnaryInfC{$Val(\top, Tr(\ulcorner \lambda \urcorner)) \vdash \lambda$}
\RightLabel{\scriptsize $\vdash \neg$}
\UnaryInfC{$\vdash \neg Val(\top, Tr(\ulcorner \lambda \urcorner)), \lambda$}
\RightLabel{\scriptsize Def of λ}
\UnaryInfC{$\vdash \lambda, \lambda$}
\RightLabel{\scriptsize Contraction}
\UnaryInfC{$\vdash \lambda$}
\end{prooftree}
$$

Call this derivation D_0.

partial negative modalities which is done within a classical setting (see [1], p.267) and when showing that they are classically paradoxical):

$$
\dfrac{\Gamma, \varphi \rightarrow \psi \vdash \Delta}{\Gamma, Val(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \vdash \Delta}\ \text{VD} \qquad\qquad \dfrac{\Gamma \vdash Val(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)\Delta}{\Gamma \vdash \varphi \rightarrow \psi, \Delta}\ \text{VD}
$$

Similarly, when Cut is available, $\vdash \varphi \rightarrow \psi$ and $\varphi \vdash \psi$ are equivalent. Thus, the following equivalence of VP can be adopted:

$$
\dfrac{\vdash \varphi \rightarrow \psi}{\vdash Val(\varphi, \psi)}\ \text{VP}
$$

Even though Cut is assumed for these equivalences, the non-transitivist might want to adopt these versions of VP and VD instead of Beall and Murzi's [5] VP and VD. No empty sequent can be reached with these equivalent rules without Cut. It is also important to note that we do not have a non-context-free VP. That is, we do not have:

$$
\dfrac{\Gamma \vdash \varphi \rightarrow \psi, \Delta}{\Gamma \vdash Val(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta}
$$

$$
\begin{prooftree}
\AxiomC{$[D_0]$}
\UnaryInfC{$\vdash \lambda$}
\AxiomC{$[D_0]$}
\UnaryInfC{$\vdash \lambda$}
\RightLabel{Transparency}
\UnaryInfC{$\vdash Tr(\ulcorner \lambda \urcorner)$}
\RightLabel{Weakening}
\UnaryInfC{$\top \vdash Tr(\ulcorner \lambda \urcorner)$}
\RightLabel{VP}
\UnaryInfC{$\vdash Val(\top, Tr(\ulcorner \lambda \urcorner))$}
\RightLabel{$\neg\vdash$}
\UnaryInfC{$\neg Val(\top, Tr(\ulcorner \lambda \urcorner)) \vdash$}
\RightLabel{Def of λ}
\UnaryInfC{$\lambda \vdash$}
\RightLabel{Cut}
\BinaryInfC{$\vdash$}
\end{prooftree}
$$

As for the higher-level Curry, $(\top \vdash_0 Tr(\ulcorner \gamma \urcorner)) \to \varphi$, it can be translated into $Val(\top, Tr(\ulcorner \gamma \urcorner)) \to \varphi$, and $Val(\top, Tr(\ulcorner \urcorner)) \to \varphi$, for any φ, is also a partial negative modality:

$$
\begin{prooftree}
\AxiomC{$\vdash \top$}
\AxiomC{$\gamma \vdash$}
\RightLabel{Transparency}
\UnaryInfC{$Tr(\ulcorner \gamma \urcorner) \vdash$}
\RightLabel{$\to\vdash$}
\BinaryInfC{$\top \to Tr(\ulcorner \gamma \urcorner) \vdash$}
\RightLabel{VD}
\UnaryInfC{$Val(\top, Tr(\ulcorner \gamma \urcorner)) \vdash$}
\RightLabel{Weakening}
\UnaryInfC{$Val(\top, Tr(\ulcorner \gamma \urcorner)) \vdash \varphi$}
\RightLabel{$\vdash\to$}
\UnaryInfC{$\vdash Val(\top, Tr(\ulcorner \gamma \urcorner)) \to \varphi$}
\end{prooftree}
$$

Let γ be its own partial negative modality $Val(\top, Tr(\ulcorner \gamma \urcorner)) \to \varphi$, we can mimic the higher-level Curry:

$$
\begin{prooftree}
\AxiomC{$\vdash \top$}
\AxiomC{$\gamma \vdash \gamma$}
\RightLabel{Transparency}
\UnaryInfC{$Tr(\ulcorner \gamma \urcorner) \vdash \gamma$}
\RightLabel{$\to\vdash$}
\BinaryInfC{$\top \to Tr(\ulcorner \gamma \urcorner) \vdash \gamma$}
\RightLabel{VD}
\UnaryInfC{$Val(\top, Tr(\ulcorner \gamma \urcorner)) \vdash \gamma$}
\RightLabel{Weakening}
\UnaryInfC{$Val(\top, Tr(\ulcorner \gamma \urcorner)) \vdash \varphi, \gamma$}
\RightLabel{$\vdash\to$}
\UnaryInfC{$\vdash Val(\top, Tr(\ulcorner \gamma \urcorner)) \to \varphi, \gamma$}
\RightLabel{Def of γ}
\UnaryInfC{$\vdash \gamma, \gamma$}
\RightLabel{Contraction}
\UnaryInfC{$\vdash \gamma$}
\end{prooftree}
$$

Call this derivation D_0.

$$
\dfrac{\dfrac{[D_0]}{\vdash \gamma} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[D_0]}{\vdash \gamma}}{\vdash Tr(\ulcorner \gamma \urcorner)}\,\text{Transparency}}{\top \vdash Tr(\ulcorner \gamma \urcorner)}\,\text{Weakening}}{\vdash Val(\top, Tr(\ulcorner \gamma \urcorner))}\,\text{VP} \qquad \varphi \vdash \varphi}{Val(\top, Tr(\ulcorner \gamma \urcorner)) \to \varphi \vdash \varphi}\,{\to}{\vdash}}{\gamma \vdash \varphi}\,\text{Def of } \gamma}{\vdash \varphi}\,\text{Cut}
$$

As for Porter's argument, our response remains the same; there is no need to ascend to higher levels. That is, we do not need VP_2, VD_2, Cut_2, and $\mathrm{Contraction}_2$ to construct such paradox. After all, the sentence that Porter provides is defined by its own partial negative modality. Thus, our recipe suggests that it is a paradox of the object language. Let us start by seeing why $\vdash Val(Val(\top, \ulcorner \urcorner), Val(\top, \bot))$ is a partial negative modality:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdash \top \qquad \pi \vdash}{\top \to \pi \vdash}\,{\to}{\vdash}}{Val(\top, \ulcorner \pi \urcorner) \vdash}\,\text{VD}}{Val(\top, \ulcorner \pi \urcorner) \vdash Val(\top, \bot)}\,\text{Weakening}}{\vdash Val(Val(\top, \ulcorner \pi \urcorner), Val(\top, \bot))}\,\text{VP}
$$

Note that just like any Curry or Validity Curry, we can put anything we want when we are using the weakening rule, and hence instead of proving $\vdash Val(\top, \bot)$ out of thin air, we could prove $\vdash \bot$ directly. Nevertheless, we will keep it to match Porter's sentence. Now, let π be its own partial negative modality $Val(Val(\top, \ulcorner \pi \urcorner), Val(\top, \bot))$. We can construct the meta-validity Curry as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \top \qquad \pi \vdash \pi}{\top \to \pi \vdash \pi}\,{\to}{\vdash}}{Val(\top, \ulcorner \pi \urcorner) \vdash \pi}\,\text{VD}}{Val(\top, \ulcorner \pi \urcorner) \vdash Val(\top, \bot), \pi}\,\text{Weakening}}{\vdash Val(\top, \ulcorner \pi \urcorner) \to Val(\top, \bot), \pi}\,{\vdash}{\to}}{\vdash Val(\top, \ulcorner \pi \urcorner) \to Val(\top, \bot), Val(Val(\top, \ulcorner \pi \urcorner), Val(\top, \bot))}\,\text{Def of } \pi}{\vdash Val(\top, \ulcorner \pi \urcorner) \to Val(\top, \bot), Val(\top, \ulcorner \pi \urcorner) \to Val(\top, \bot)}\,\text{VD}}{\vdash Val(\top, \ulcorner \pi \urcorner) \to Val(\top, \bot)}\,\text{Contraction}}{\vdash Val(Val(\top, \ulcorner \pi \urcorner), Val(\top, \bot))}\,\text{VP}}{\vdash \pi}\,\text{Def of } \pi
$$

Call this Derivation D_0.

$$
\begin{prooftree}
\AxiomC{$[D_0]$}
\UnaryInfC{$\vdash \pi$}
\AxiomC{$[D_0]$}
\UnaryInfC{$\vdash \pi$}
\RightLabel{Weakening}
\UnaryInfC{$\top \vdash \pi$}
\RightLabel{VP}
\UnaryInfC{$\vdash Val(\top, \ulcorner \pi \urcorner)$}
\AxiomC{$Val(\top, \bot) \vdash Val(\top, \bot)$}
\RightLabel{$\to\vdash$}
\BinaryInfC{$Val(\top, \ulcorner \pi \urcorner) \to Val(\top, \bot) \vdash Val(\top, \bot)$}
\RightLabel{VD}
\UnaryInfC{$Val(Val(\top, \ulcorner \pi \urcorner), Val(\top, \bot)) \vdash Val(\top, \bot)$}
\RightLabel{Def of π}
\UnaryInfC{$\pi \vdash Val(\top, \bot)$}
\RightLabel{Cut}
\BinaryInfC{$\vdash Val(\top, \bot)$}
\end{prooftree}
$$

This shows that the substructuralist does not need to ascend to metainferences to construct higher-level paradoxes. The validity predicate is powerful enough to creep these paradoxes in the inferential level. In other words, these higher-level paradoxes collapse into validity paradoxes. After all, the validity predicate does not only express valid inferences, but it also expresses valid meta$_n$inferences in the object language.

As a result, the substructuralist still provides a uniform solution to semantic paradoxes, including higher-level paradoxes, since there is no need for higher-level rules.

2.1 Dissolving the Hierarchy Further

One might argue that what we have been calling 'additional machinery' already exists in the system; we are not *adding* any new rules to the system, but rather, we are just labeling them. For example, the system already obeys meta-Cut. It just has not been labeled.[7]

It seems, prima facie, that something new is being added when we include names to sequents or when we add operational and structural rules that apply to sequents rather than formulas. However, for the sake of argument, let us suppose that we are merely labeling pre-existing and presupposed rules. We claim that even if this is the case, there is a reading available for the substructuralists that allows them to dissolve the hierarchy altogether: We simply read every '$\vdash$' and '$\to$' as '$\Rightarrow$', and commas before the turnstiles are read as conjunctions and commas after the turnstiles are read as disjunctions. In other words, conditionals and turnstiles are treated the same.[8] Now, we can transform our rules such that every '$\vdash$' and '$\to$' turns into a '$\Rightarrow$'. For instance, the transformation of the ($\vdash\to$) rule would be:

[7]We owe Brian Porter for this pushback (via personal communication on October 17th, 2021).

[8]If we think that the main difference between conditionals and turnstiles is use vs. mention respectively, then in this case, they both play the "use" role while the validity predicate plays the "mention" role.

$$\frac{\Gamma, \mathcal{A} \vdash \mathcal{B}, \Delta}{\Gamma \vdash \mathcal{A} \to \mathcal{B}, \Delta} \quad \rightsquigarrow \quad \frac{(\bigwedge \Gamma \wedge \mathcal{A}) \Rightarrow (\mathcal{B} \vee \bigvee \Delta)}{\bigwedge \Gamma \Rightarrow ((\mathcal{A} \Rightarrow \mathcal{B}) \vee \bigvee \Delta)}$$

The horizontal line is just another '$\Rightarrow$'. Hence, we can read the previous transformation as:

$$((\bigwedge \Gamma \wedge \mathcal{A}) \Rightarrow (\mathcal{B} \vee \bigvee \Delta)) \Rightarrow (\bigwedge \Gamma \Rightarrow ((\mathcal{A} \Rightarrow \mathcal{B}) \vee \bigvee \Delta))$$

We transform all of the other rules in a similar fashion. Therefore, the Cut rule would be transformed as follows:

$$\frac{\Gamma \vdash \mathcal{A}, \Delta \qquad \Gamma, \mathcal{A} \vdash \Delta}{\Gamma \vdash \Delta} \quad \rightsquigarrow \quad \frac{(\bigwedge \Gamma \Rightarrow \mathcal{A} \vee \bigvee \Delta)) \wedge (\bigwedge \Gamma \wedge \mathcal{A} \Rightarrow \bigvee \Delta)}{\bigwedge \Gamma \Rightarrow \bigvee \Delta}$$

While the meta-Cut rule:[9]

From

$$\frac{\Gamma \vdash \Delta}{\Theta \vdash \Xi} \quad \text{and} \quad \frac{\Theta \vdash \Xi}{\Sigma \vdash \Pi}$$

we infer

$$\frac{\Gamma \vdash \Delta}{\Sigma \vdash \Pi}$$

would transform as follows:

$$\frac{(\bigwedge \Gamma \Rightarrow \bigvee \Delta) \Rightarrow (\bigwedge \Theta \Rightarrow \bigvee \Xi)) \wedge (\bigwedge \Theta \Rightarrow \bigvee \Xi) \Rightarrow (\bigwedge \Sigma \Rightarrow \bigvee \Pi)}{(\bigwedge \Gamma \Rightarrow \bigvee \Delta) \Rightarrow (\bigwedge \Sigma \Rightarrow \bigvee \Pi)}$$

Notice that this renders meta-Cut as an instance of our regular Cut; the Cut-formula $\mathcal{A}$ is instantiated by $\bigwedge \Theta \Rightarrow \bigvee \Xi$. This is akin to using Cut on a conditional.

The same phenomenon occurs with Contraction; meta-Contraction becomes an instance of Contraction. Here is the transformation of Contraction:

$$\frac{\Gamma, \mathcal{A}, \mathcal{A} \vdash \Delta}{\Gamma, \mathcal{A} \vdash \Delta} \quad \rightsquigarrow \quad \frac{(\bigwedge \Gamma \wedge \mathcal{A} \wedge \mathcal{A}) \Rightarrow \bigvee \Delta}{(\bigwedge \Gamma \wedge \mathcal{A} \Rightarrow) \bigvee \Delta}$$

Right Contraction is transformed in a similar fashion. As for meta-Contraction:

From

[9] This is a simplified version of the rule to make it more legible, but the process is the same.

$$\frac{\Gamma \vdash \Delta \qquad \Gamma \vdash \Delta}{\Sigma \vdash \Pi}$$

we infer

$$\frac{\Gamma \vdash \Delta}{\Sigma \vdash \Pi}$$

Its transformation would be as follows:

$$\frac{((\Gamma \Rightarrow \Delta) \wedge (\Gamma \Rightarrow \Delta)) \Rightarrow (\Sigma \Rightarrow \Pi)}{((\Gamma \Rightarrow \Delta)) \Rightarrow (\Sigma \Rightarrow \Pi)}$$

Right meta-Contraction would be transformed in a similar fashion. Again, this renders meta-Contraction to be an instance of Contraction; we are simply contracting conditionals.

Thus, there is a reading available for the substructuralist such that targeting Cut would be enough to solve all semantic paradoxes, including higher-level paradoxes, since it would automatically target every meta$_n$Cut. Similarly, targeting Contraction would automatically target every meta$_n$Contraction. Therefore, the substructuralist maintains uniformity. To clarify, we are not saying that that is how we *should* read these (meta)sequents, but rather a weaker claim—that this a possible direction one could take to further dissolve the hierarchy.

One might claim that this approach of collapsing the hierarchy motivates collapsing down to an operational level. Thus, a non-transitive approach would collapse to the logic *LP*, so this would be an argument against the substructuralist.[10] However, there are good reasons for ascending one level higher than *LP* (i.e., the level of *ST*). For example, it adheres to the principle of minimal mutilation to classical logic while maintaining a uniform solution to semantic paradoxes.

An immediate response to this is that if our objective is to minimally mutilate classical logic while providing a uniform solution to semantic paradoxes, then this would force us to ascend the *ST*-hierarchy. After all, each level of the hierarchy provides a uniform solution to semantic paradoxes while getting us closer and closer to classical logic (see, [14] and [21]), and certainly, we can collapse the hierarchy to our chosen level. Nevertheless, minimal mutilation to classical logic is not the only logical virtue to consider. For instance, even though every level that you ascend to would get you closer to classical logic, this is done on the expense of simplicity, and we take simplicity to be a logical virtue. Thus, logical virtues should not be considered in isolation, since achieving them

[10] Many thanks to the anonymous reviewer for this possible objection against the substructuralist. Also, thanks to Federico Pailos for raising a similar concern (via personal communications on November 23rd, 2023).

might cost us to forgo other virtues. For further arguments against ascending the ST-hierarchy, see Ripley's [19], Hlobil's [12], and Scambler's [21].[11]

3 Validity, Admissibility, and Internalization

We claimed earlier that we do not need to ascend to higher levels since the validity predicate is powerful enough to not only creep the higher-level paradoxes in the object level, but also to express valid inferences and meta$_n$inferences. However, one might worry that the validity predicate cannot express everything we expect it to express. For instance, Barrio, Rosenblatt, and Tajer claim that the non-transitive logic ST (plus the validity predicate) cannot internalize some of its metarules ([2], p.713). They support their claim by showing, as an example, that a simplified version of the metarule $(\neg\vdash)$,

$$\frac{\varphi \vdash \psi, \chi}{\varphi, \neg\chi \vdash \psi}$$

cannot be internalized; it cannot be expressed by the validity predicate. They take the internalization of the aforementioned meta-rule to be $\vdash Val(\varphi, \psi \vee \chi) \rightarrow Val(\varphi \wedge \neg\chi, \psi)$,[12] and that sequent is not provable in a Cut-free system plus the validity predicate with its generalized rules VP' and VD':

$$\frac{}{\Gamma, Val(\bigwedge\Gamma, \bigvee\Delta) \vdash \Delta}\text{ VD'} \qquad \frac{\Gamma \vdash \Delta}{\vdash Val(\bigwedge\Gamma, \bigvee\Delta)}\text{ VP'}$$

They show the following attempt and note how the conclusion is not reachable:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Val(\varphi, \psi \vee \chi), \varphi \vdash \psi, \chi}\text{ VD'}}{Val(\varphi, \psi \vee \chi), \varphi, \neg\chi \vdash \psi}\ \neg\vdash}{Val(\varphi, \psi \vee \chi), \varphi \wedge \neg\chi \vdash \psi}\ \wedge\vdash}{Val(\varphi, \psi \vee \chi) \vdash Val(\varphi \wedge \neg\chi, \psi)}\ ??$$

Concluding that in order to prove that last step, VP' must be non-context-free ([2], p.713). Alternatively, we can strengthen both VP' and VD' in order to internalize its primitive and derivable metarules ([2], p.716-718).

[11] Scambler [21] does not only argue against ascending the ST-hierarchy, but also argues against ascending to ST. That is, Scambler is arguing for LP. In [16], Porter argues that Scambler's argument against the ST-hierarchy does not work against ST, but it could *potentially* be a problem for advocates of ST.

[12] For more on the notion of internalization, see [20].

However, the final sequent is not reachable even in the presence of Cut. A metarule, in its general form, is a matter of admissibility. In terms of validity, the Cut-free logic can internalize all valid forms of these metarules. For example, this derivation

$$\frac{\varphi \vdash \phi, \chi}{\varphi, \neg\chi \vdash \phi}$$

can be internalized the way Barrio et al. suggested—$Val(\varphi, \varphi \vee \chi) \vdash Val(\varphi \wedge \neg\chi, \varphi)$. This is done by proving the succedent first, and then we can use weakening for the antecedent.

In order to show how we can internalize the metarules in their general forms, let us first state the rules for our admissibility predicate. It is safe to say that the admissibility predicate governs the following two rules:[13] [14]

(A1) If $\alpha \rightarrow \beta \vdash \gamma \rightarrow \delta$ then $\vdash Adm(\ulcorner \alpha \rightarrow \beta \urcorner, \ulcorner \gamma \rightarrow \delta \urcorner)$

(A2) $Adm(\ulcorner \alpha \rightarrow \beta \urcorner, \ulcorner \gamma \rightarrow \delta \urcorner), \alpha \rightarrow \beta \vdash \gamma \rightarrow \delta$

We can now internalize the metarule Barrio et al. have mentioned:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi \qquad \dfrac{\psi \vdash \psi \qquad \chi \vdash \chi}{\psi \vee \chi \vdash \psi, \chi}\vee\vdash}{\varphi, \varphi \rightarrow (\psi \vee \chi) \vdash \psi, \chi}\rightarrow\vdash}{\varphi, \neg\chi, \varphi \rightarrow (\psi \vee \chi) \vdash \psi}\neg\vdash}{\varphi \wedge \neg\chi, \varphi \rightarrow (\psi \vee \chi) \vdash \psi}\wedge\vdash}{\varphi \rightarrow (\psi \vee \chi) \vdash (\varphi \wedge \neg\chi) \rightarrow \psi}\vdash\rightarrow}{\vdash Adm(\ulcorner \varphi \rightarrow (\psi \vee \chi) \urcorner, \ulcorner (\varphi \wedge \neg\chi) \rightarrow \psi \urcorner)}\text{A1}$$

However, admissibility and validity are related. After all, we take the admissibility of a metarule, say,

$$\frac{\Gamma \vdash \Delta}{\Sigma \vdash \Xi}$$

to mean that given that $\Gamma \vdash \Delta$ holds, then so does $\Sigma \vdash \Xi$, and 'holds' here means nothing but our good old friend validity. So, we might still want to

[13] Alternatively, we can have names for sequents (see [10]), or even a four-place admissibility predicate.

[14] Given that the admissibility predicate is partially transparent, there will, of course, be some paradoxical sentences that use the admissibility predicate (See [1], [10]).

internalize the metarules in terms of the validity predicate. We will now turn to two possible internalizations that the non-transitivist could adopt without needing to strengthen VP or VD (or their generalized forms: VP' and VD').

First Internalization

If we want to express the admissibility of metarules via the validity predicate without strengthening VP and VD, then we can internalize the aforementioned metarule, $\{\Gamma \vdash_0 \Delta\} \vdash_1 \{\Sigma \vdash_0 \Xi\}$, as $Val(\bigwedge \Gamma \rightarrow \bigvee \Delta, \bigwedge \Sigma \rightarrow \bigvee \Xi)$ rather than $Val(\Gamma, \Delta) \vdash Val(\Sigma, \Xi)$. Here, we used conditionals instead of sequents or validity predicates because sequents and validity predicates are just metatheoretic conditionals. So instead of introducing names for sequents, we just represent them as conditionals. Barrio et al.'s internalization represents the '$\vdash_0$'s as validity predicates while they represent '$\vdash_1$' as $\vdash_0$ (or equivalently, a conditional). We simply flipped that; we represented the '$\vdash_0$'s as conditionals and the '$\vdash_1$' as a validity predicate. Thus, it is more in line with the level of the sequents. Moreover, this internalization would allow us to internalize the metarules without any disturbance to our VP and VD rules (or their generalized forms). Primarily, we do not want to strengthen VP to a non-context free version.

It is not hard to see that we can now internalize the metarules of our logic without the need to strengthen VP' or VD' as Barrio et al. suggest. For example, the simplified version of $(\neg\vdash)$ can now be internalized as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi \qquad \dfrac{\psi \vdash \psi \qquad \chi \vdash \chi}{\psi \vee \chi \vdash \psi, \chi}\vee\vdash}{\varphi, \varphi \rightarrow (\psi \vee \chi) \vdash \psi, \chi}\rightarrow\vdash}{\varphi, \neg\chi, \varphi \rightarrow (\psi \vee \chi) \vdash \psi}\neg\vdash}{\varphi \wedge \neg\chi, \varphi \rightarrow (\psi \vee \chi) \vdash \psi}\wedge\vdash}{\varphi \rightarrow (\psi \vee \chi) \vdash (\varphi \wedge \neg\chi) \rightarrow \psi}\vdash\rightarrow}{\vdash Val(\ulcorner \varphi \rightarrow (\psi \vee \chi)\urcorner, \ulcorner (\varphi \wedge \neg\chi) \rightarrow \psi \urcorner)}\text{VP}$$

Note that now the validity predicate is behaving exactly as our admissibility predicate, but this should not come as a surprise. After all, we are trying to express admissibility via the validity predicate.[15]

Whether we represent the metarules with the admissibility predicate or the validity predicate, there is still a pressing issue; the problem raised by Barrio et al. ([2], p.718) affects us as well. The problem goes as follows: Even though we cannot reach the empty sequent in validity Curry, it can still internalize

[15] For a different response to Barrio et al's internalization problem, see Golan's [9]. Golan claims that ST does not need to internalize its metarules using the object-language validity predicate; that is a job for a higher-level validity predicate (e.g., Val_1 should be the predicate that captures the validity of metarules). Each validity predicate captures one level of (meta)inference.

an inadmissible instance of Cut. Take your regular validity Curry sentence, v, where it is equivalent to $Val(v, p)$:

$$
\begin{prooftree}
\AxiomC{$\vdash \top$}
\AxiomC{}
\RightLabel{VD}
\UnaryInfC{$v, Val(v, p) \vdash p$}
\RightLabel{Def of v}
\UnaryInfC{$v, v \vdash p$}
\RightLabel{Contraction}
\UnaryInfC{$v \vdash p$}
\RightLabel{$\to\vdash$}
\BinaryInfC{$\top \to v \vdash p$}
\RightLabel{Weakening}
\UnaryInfC{$\top \to v, \top \vdash p$}
\RightLabel{$\vdash\to$}
\UnaryInfC{$\top \to v \vdash \top \to p$}
\RightLabel{Weakening}
\UnaryInfC{$\top \to v, v \to p \vdash \top \to p$}
\RightLabel{$\wedge\vdash$}
\UnaryInfC{$(\top \to v) \wedge (v \to p) \vdash \top \to p$}
\RightLabel{VP}
\UnaryInfC{$\vdash Val((\top \to v) \wedge (v \to p), \top \to p)$}
\end{prooftree}
$$

Similarly, let ϱ be its own partial negative modality $Adm(\ulcorner \top \to \varrho \urcorner, \ulcorner \top \to \bot \urcorner)$:[16]

$$
\begin{prooftree}
\AxiomC{$\vdash \top$}
\AxiomC{}
\RightLabel{A2}
\UnaryInfC{$Adm(\ulcorner \top \to \varrho \urcorner, \ulcorner \top \to \bot \urcorner), \top \to \varrho \vdash \top \to \bot$}
\RightLabel{Def of ϱ}
\UnaryInfC{$\varrho, \top \to \varrho \vdash \top \to \bot$}
\RightLabel{$\to\vdash$}
\BinaryInfC{$\top \to \varrho, \top \to \varrho \vdash \top \to \bot$}
\RightLabel{Contraction}
\UnaryInfC{$\top \to \varrho \vdash \top \to \bot$}
\RightLabel{A1}
\UnaryInfC{$\vdash Adm(\ulcorner \top \to \varrho \urcorner, \ulcorner \top \to \bot \urcorner)$}
\RightLabel{Def of ϱ}
\UnaryInfC{$\vdash \varrho$}
\end{prooftree}
$$

Call this derivation D_0

$$
\begin{prooftree}
\AxiomC{$[D_0]$}
\UnaryInfC{$\vdash \varrho$}
\AxiomC{$\bot \vdash$}
\RightLabel{$\to\vdash$}
\BinaryInfC{$\varrho \to \bot \vdash$}
\RightLabel{Weakening}
\UnaryInfC{$\top \to \varrho, \varrho \to \bot \vdash$}
\RightLabel{Weakening}
\UnaryInfC{$\top \to \varrho, \varrho \to \bot \vdash \top \to \bot$}
\RightLabel{$\wedge\vdash$}
\UnaryInfC{$(\top \to \varrho) \wedge (\varrho \to \bot) \vdash \top \to \bot$}
\RightLabel{A1}
\UnaryInfC{$\vdash Adm(\ulcorner (\top \to \varrho) \wedge (\varrho \to \bot) \urcorner, \ulcorner \top \to \bot \urcorner)$}
\end{prooftree}
$$

Thus, we have proved an admissibility of an unwanted version of Cut—if $\top \vdash \varrho$ and $\varrho \vdash \bot$ hold, then so does $\top \vdash \bot$. $\top \vdash \varrho$ and $\varrho \vdash \bot$ do indeed hold in a non-transitive approach, but $\top \vdash \bot$ does not.

So, whether we use the admissibility predicate, our proposed internalization, or Barrio et al's internalization (plus strengthening VP'), there are instances of unwanted internalizations. In [6], Cook argues that if we take Val to be *logical* validity, then the final VP move is unwarranted.[17] That is because the use of

[16] This is Hlobil's ACu [10].

[17] Cook was arguing against Beall and Murzi's validity-Curry's paradox in [5] (see [6], p.452-453 for Cook's argument), but the same argument applies here. Cook further argues that the

'Def of v' in the beginning of the proof depends on arithmetic, and thus, the premise of the VP move should read: $PA, (\top \to v) \wedge (v \to p) \vdash \top \to p$. So, we cannot apply VP here because it is not context-free ([6], p.453). Moreover, VD and VP are not logical rules ([6], p.465). So, an application of VP in a derivation that uses either VP, VD, or the diagonalization lemma would be a mistake if Val is taken to be logical validity.[18] Similar reasoning applies to the second argument if we take Adm to be logical admissibility. This renders the final moves in the previous arguments to consist of misapplications of VP and A1. However, the internalization of the metarules such as $(\neg\vdash)$ above would still go through; there is no misapplication of VP (or A1 in the admissibility case) since we did not rely on an arithmetical truth (e.g., the argument did not rely on the diagonalization lemma) nor did we use any non-logical rules in the derivations prior to using VP (or A1).

So, maybe we could just insist that Val has to be logical validity, and thus we would be able to internalize the metarules without internalizing the unwanted instances of Cut. Alas, this approach would not work with our proposed internalization. Given our proposed internalization, we can internalize every instance of Cut without using VD or the diagonalization lemma. That is, for any φ, ψ, and χ, we have $\vdash Val((\varphi \to \psi) \wedge (\psi \to \chi), \varphi \to \chi)$:[19]

$$
\begin{prooftree}
\AxiomC{$\varphi \vdash \varphi$}
\AxiomC{$\psi \vdash \psi$}
\RightLabel{$\to\vdash$}
\BinaryInfC{$\varphi \to \psi, \varphi \vdash \psi$}
\AxiomC{$\chi \vdash \chi$}
\RightLabel{$\to\vdash$}
\BinaryInfC{$\varphi \to \psi, \psi \to \chi, \varphi \vdash \chi$}
\RightLabel{$\vdash\to$}
\UnaryInfC{$\varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$}
\RightLabel{$\wedge\vdash$}
\UnaryInfC{$(\varphi \to \psi) \wedge (\psi \to \chi) \vdash \varphi \to \chi$}
\RightLabel{VP}
\UnaryInfC{$\vdash Val((\varphi \to \psi) \wedge (\psi \to \chi), \varphi \to \chi)$}
\end{prooftree}
$$

It follows that we can internalize an unwanted instance of Cut even if we take Val to be logical validity. This means that our internalization is behaving classically—our internalization claims that Cut is admissible in the system.

The issue of unwanted internalizations is a serious issue. However, the issue is not unique to the validity predicate. In fact, this issue occurs in other partially transparent predicates. For example, in [22], Shapiro claims that in systems similar to the one we are discussing, we can prove ϑ and $\neg\vartheta$ where ϑ is $\neg Bew(\ulcorner\vartheta\urcorner)$. Since ϑ is provable and $Bew(y)$ stands for $\exists x Prf(x, y)$, we can prove $\exists x Prf(x, \ulcorner\vartheta\urcorner)$ (i.e., there is a proof for ϑ). However, ϑ is equivalent to $\forall x \neg Prf(x, \ulcorner\vartheta\urcorner)$. Hence, we have a proof that there is no proof of ϑ, yet we have a proof of ϑ. It is tempting to conclude that $\forall x \neg Prf(x, \ulcorner\vartheta\urcorner)$ is false, but

culprit of the validity-Curry is the assumption that VP and VD are logically valid; we should not use VP on a subderivation that uses VD or VP since they are not logically valid rules, and hence, there are no paradoxes of logical validity ([6], p.459-465). For related arguments, see [7,13].

[18] Note that Barrio et al. claim that they are not taking validity to be a purely logical concept ([2], p.709 in fn.).

[19] Many thanks to the anonymous reviewer for this observation.

it is not. That sentence is ungrounded. This is similar to how tempting it is to call the Liar false, but in fact it is ungrounded.

Barrio et al's criticism is just another version of Shapiro's. The validity Curry and the admissibility Curry lie to you and tell you that they are cuttable. It is tempting to say it is false, but it is ungrounded. Another version of this comes by looking at the validity liar λ from earlier. We can ignore the truth predicate, so λ is $\neg Val(\top, \lambda)$. Since we can prove λ, we have a proof that $\vdash \lambda$ is not valid, but we have a proof of it, so it should be valid. Indeed, we can also show that it is valid. That is the nature of paradoxical ungrounded sentences; they lie to you. Just as how we cannot treat ungrounded sentences as false, we cannot treat them as true either.

However, there is a serious disanalogy here; the unwanted internalizations of Cut can be achieved without appealing to sentences such as v.[20] Thus, we cannot blame v for the unwanted internalization. However, as we will see in the second internalization, the admissibility of Cut is not internalizable, but it still internalizes the claim that v is cuttable. So, in the second internalization (and in Barrio et al.'s internalization), v is doing the dirty work—v is lying to us.

In light of the overinternalization problem, Hlobil [11] suggests that the non-transitivist must reject VD. If we reject VD and generalize VP (via allowing the logic to assume and discharge sequents), then the validity predicate is faithful—internalizes all and only the derivable meta-rules ([11], p.15). Such an approach solves the dilemma, but presents a further problem—non-uniformity. Hlobil addresses this problem and claims that "it doesn't seem problematic to blame the v-Curry on VD, while blaming Cut for the other paradoxes. For neither faithfulness nor VD play a role in any of the other paradoxes" ([11], p.10). This, however, is not true; other partially transparent predicates face the issue of faithfulness, as we have just shown. Also, as we have argued in [1], VD is not special, and every partially transparent predicate has a rule similar to VD. Whether it is a one-place or an n-place predicate is irrelevant. Hence, to maintain uniformity, Hlobil's approach entails that we should require faithfulness for other predicates and reject every rule that is similar to VD. Such an approach, though possible, is not ideal because it would require a justification for rejecting each of these rules ([1], p.278). Moreover, we would not need to reject Cut to avoid the paradoxes if we reject these rules. The paradoxical sentences would not be provable, hence, there is no threat in reaching the empty sequent (or an unwanted sequent) via Cut.

Of course, we are not saying that there can be no ground for treating two paradoxes differently. Rather, we are claiming that such an approach would require a lot more work compared to other more uniform solutions. Additionally, if the validity paradoxes should be treated differently from other semantic para-

[20] Thanks to the anonymous reviewer for pointing out this disanalogy.

doxes, then it cannot be motivated by the reasons Hlobil has suggested since the issue of faithfulness and rules parallel to VD are present in other semantic predicates.

Second Internalization

We can, alternatively, internalize a metarule, say,

$$\dfrac{\Gamma \vdash \Delta}{\Sigma \vdash \Xi}$$

as $Val(Val(\Gamma, \Delta), \bigwedge \Sigma \to \bigvee \Xi)$. Here, we retain the idea that $\Sigma \vdash \Xi$ is valid *from the validity of* $\Gamma \vdash \Delta$. [21] In this alternative internalization, the simplified version of $(\neg \vdash)$ can be internalized as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Val(\varphi, \psi \lor \chi), \varphi \vdash \psi, \chi}\ \text{VD'}}{Val(\varphi, \psi \lor \chi), \varphi, \neg\chi \vdash \psi}\ \neg\vdash}{Val(\varphi, \psi \lor \chi), \varphi \land \neg\chi \vdash \psi}\ \land\vdash}{Val(\varphi, \psi \lor \chi) \vdash (\varphi \land \neg\chi) \to \psi}\ \vdash\to}{\vdash Val(Val(\varphi, \psi \lor \chi), (\varphi \land \neg\chi) \to \psi)}\ \text{VP'}$$

We can internalize all the metarules in a similar fashion; we start with VD' (or two VD' if we have a metarule with two upper sequents), followed by an application of a rule we would like to internalize. As a result, we cannot internalize a general version of Cut unless we can actually perform Cut:[22]

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Val(\varphi, \psi), \varphi \vdash \psi}\ \text{VD'} \qquad \dfrac{}{Val(\psi, \chi), \psi \vdash \chi}\ \text{VD'}}{Val(\varphi, \psi), Val(\psi, \chi), \varphi \vdash \chi}\ \text{Cut}}{Val(\varphi, \psi), Val(\psi, \chi) \vdash \varphi \to \chi}\ \vdash\to}{\vdash Val(Val(\varphi, \psi) \land Val(\psi, \chi), \varphi \to \chi)}\ \text{VP'}$$

[21] The motivation for this internalization becomes clearer when we consider the invalidities of metainferences and compare it with Barrio et al.'s internalization. Suppose we want to say that the sequent $\Sigma \vdash \Xi$ is not valid from the validity of $\Gamma \vdash \Delta$. Given how Barrio et al. internalize validity, they would have to internalize the previous sentence as $Val(\Gamma, \Delta) \to \neg Val(\Sigma, \Xi)$. Indeed, that is how they do it. They claim that "we can also 'internalize' the claim that Cut does *not* hold in the instance involving π, in the sense that $[\vdash] Val(\top, \pi) \land Val(\pi, \bot) \to \neg Val(\top, \bot)$" ([2], p.718 in fn.). However, it is very possible that $\vdash \bot$ does not follow from $\top \vdash \pi$ and $\pi \vdash \bot$, but $\vdash \bot$ might still be derivable by other means. In such a case, $(Val(\top, \pi) \land Val(\pi, \bot)) \to \neg Val(\top, \bot)$ would come out as false even though it is true that Cut does not hold for this instance. If we want to express that Cut does not hold, for an instance such as the one above, in our second proposed internalization, then we can do so with the following: $\neg Val(Val(\top, \pi) \land Val(\pi, \bot), \top \to \bot)$. That way, that sentence is true even if $\vdash \bot$ is derivable by other means.

[22] Note that the non-transitivist must reject the equivalences of VD mentioned in footnote 6 in this internalization. Otherwise, Cut would be admissible—the same way it is admissible in our first proposed internalization.

Does that mean that there are no unwanted internalizations of Cut in this alternative internalization? Unfortunately, they still appear in this internalization. Consider v once more:[23]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Val(v,p), v \vdash p}\ \text{VD'}}{Val(v,p), Val(v,p) \vdash p}\ \text{def of } v}{Val(v,p) \vdash p}\ \text{Contraction}}{Val(v,p), \top \vdash p}\ \text{Weakening}}{Val(v,p) \vdash \top \to p}\ \vdash\to}{Val(\top, v), Val(v,p) \vdash \top \to p}\ \text{Weakening}}{\vdash Val(Val(\top, v) \wedge Val(v,p), \top \to p)}\ \text{VP'}$$

Notice that this alternative internalization (plus VP' and VD') is behaving a lot like Barrio et al.'s internalization (plus their VP$^+$ and VD$^+$). The main difference is that our alternative internalization would keep our original VP and VD rules (or their general forms, VP' and VD'), and would insist on a context-free VP.[24]

Furthermore, an application of the diagonalization lemma appears before applying VP'. Therefore, a possible way to overcome this overinternalization is to insist that the final VP' move is unwarranted since, technically speaking, we are performing a non-context free VP' given that a diagonalization lemma was used (see [6], p.451-453). This response is also available for Barrio et al.'s internalization (plus their VP$^+$ and VD$^+$).

One more thing to point out about this alternative internalization is that VD' is required to internalize our metarules, and according to Cook [6], this means that our Val is not a notion of logical validity. However, Cook only shows that VP is not logically valid because it does not satisfy logical substitutivity ([6], p.460-461). It is not clear how we would show that VD does not satisfy logical substitutivity. Hence, there is still some room for arguing that even though VP is not logically valid, VD might be logically valid. If such an argument is successful, then our Val here would reflect the notion of logical validity.

Let us sum up the options presented in this paper (these options are by no means exhaustive):

- Option 1 (Barrio et al.'s proposal): Admit that Barrio et al.'s internalization is correct, but reject VP and VD and replace them with stronger versions such as VP$^+$ and VD$^+$. This would allow us to internalize the

[23] As mentioned earlier, v is the culprit for this unwanted internalization. Thus, unlike our first proposed internalization, we can blame v for this result.

[24] Even though Barrio et al's VP$^+$ is non-context-free, the context is still restricted to validities.

metarules, but it would also admit internalizations of unwanted instances of Cut. As a result, we can either reject these overinternalizations and claim that they contain misapplications of VP^+ since they rely on the diagonalization lemma that adds extra contexts that are not of the form of validities, or accept the problem of overinternalization and point out that this problem is present in other partially transparent predicates.

- Option 2 (Hlobil's proposal): Admit that Barrio et al.'s internalization is correct, reject VD, and generalize VP by allowing the logic to assume and discharge sequents. This allows us to internalize the metarules without internalizing unwanted instances of Cut. However, this would suggest a non-uniform solution to semantic paradoxes, and the reasons Hlobil provides for rejecting VD are not enough because the issue of faithfulness and rules similar to VD are still present in other partially transparent predicates. If someone wants to take this route, then further arguments for rejecting VD are needed.

- Option 3 (Our first proposal): Keep VP and VD (or their generalized form) as they are, but replace Barrio et al.'s internalization with an internalization that keeps the levels of the sequents intact. That way, we can internalize our metarules. However, all instances of Cut (whether wanted or not) are internalizable even if we take Val to be a notion of logical validity.

- Option 4 (Our second proposal): Keep VP and VD (or their generalized form) as they are, but replace Barrio et al.'s internalization with the alternative internalization that we proposed. This allows us to internalize the metarules without internalizing every instance of Cut. However, it still admits some unwanted internalizations of Cut. As a response, we can reject these unwanted internalizations by pointing out that they rely on the diagonalization lemma, and hence the final VP move is unwarranted since it is not context-free. Alternatively, we make peace with the problem and accept that this is bound to happen to all partially transparent predicates.

Unfortunately, there is no clear winner in these proposed options. If VD is shown to be not logically valid, and we want Val to be a notion of logical validity, then our only option is option 3. Option 1 and option 4 are in a similar situation, but we must choose whether we want to keep the rules VP and VD or keep Barrio et al.'s internalization. Finally, option 2 might become more appealing than the other options if we find better grounds for rejecting VD.

4 Conclusion

We addressed the recent charges that claim that the substructural solutions are not as uniform as they appear to be. We showed that these higher-level paradoxes are just validity paradoxes wearing oversized clothing. It is not clear that there are any reasons that force the substructuralist to expand their language to include names for sequents and higher-level rules. We then addressed the worry that the validity predicate is not strong enough to internalize the metarules of our logic. We explored two options that can internalize the metarules without the need to strengthen the validity predicate. However, we showed that there are still unwanted internalizations of Cut. Finally, we listed some pros and cons of the options discussed in this paper, and possible responses to the overinternalization problem.[25]

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References

[1] Ahmad, R. A Recipe for Paradox. *The Australasian Journal of Logic 19*, 5 (Dec. 2022), 254–281.

[2] Barrio, E., Rosenblatt, L., and Tajer, D. Capturing Naive Validity in the Cut-free Approach. *Synthese 199*, S3 (2016), 707–723.

[3] Barrio, E. A., Pailos, F., and Szmuc, D. (Meta)Inferential Levels of Entailment Beyond the Tarskian Paradigm. *Synthese 198*, S22 (2019), 5265–5289.

[4] Barrio, E. A., Pailos, F., and Szmuc, D. A Hierarchy of Classical and Paraconsistent Logics. *Journal of Philosophical Logic 49*, 1 (Feb. 2020), 93–120.

[5] Beall, J., and Murzi, J. Two Flavors of Curry's Paradox. *Journal of Philosophy 110*, 3 (2013), 143–165.

[6] Cook, R. T. There is No Paradox of Logical Validity. *Logica Universalis 8*, 3 (Jan. 2014), 447–467.

[7] Field, H. Disarming a Paradox of Validity. *Notre Dame Journal of Formal Logic 58*, 1 (Jan. 2017).

[8] French, R. Structural Reflexivity and the Paradoxes of Self-Reference. *Ergo, an Open Access Journal of Philosophy 3*, 20201214 (Apr. 2016).

[9] Golan, R. Metainferences from a Proof-Theoretic Perspective, and a Hierarchy of Validity Predicates. *Journal of Philosophical Logic* (July 2021), 1–31.

[10] Hlobil, U. The Cut-Free Approach and the Admissibility-Curry. *Thought: A Journal of Philosophy 7*, 1 (2018), 40–48.

[11] Hlobil, U. Faithfulness for Naive Validity. *Synthese 196*, 11 (Nov. 2019), 4759–4774.

[12] Hlobil, U. A truth-maker semantics for ST: Refusing to climb the strict/tolerant hierarchy. *Synthese 200*, 5 (Aug. 2022), 368.

[13] Ketland, J. Validity as a Primitive. *Analysis 72*, 3 (2012), 421–430.

[14] Pailos, F. M. A Family of Metainferential Logics. *Journal of Applied Non-Classical Logics 29*, 1 (Jan. 2019), 97–120.

[15] Pailos, F. M. A Fully Classical Truth Theory Charactarized by Substructural Means. *The Review of Symbolic Logic 13*, 2 (June 2020), 249–268.

[16] Porter, B. Supervaluations and the Strict-Tolerant Hierarchy. *Journal of Philosophical Logic 51*, 6 (July 2021), 1367–1386.

[17] Porter, B. C. *Three Essays on Substructural Approaches to Semantic Paradoxes*. PhD thesis, City University of New York (CUNY), 2023.

[18] Priest, G. Substructural Solutions to the Semantic Paradoxes: A Dialethic Perspective. In *Paradoxes Between Truth and Proof*, M. Petrolo and G. Venturi, Eds. Synthese Library, Springer, Forthcoming.

[19] Ripley, D. One Step is Enough. *Journal of Philosophical Logic 51*, 6 (Dec. 2022), 1233–1259.

[20] Rosenblatt, L. Naive Validity, Internalization, and Substructural Approaches to Paradox. *Ergo, an Open Access Journal of Philosophy 4*, 4 (Apr. 2017).

[21] Scambler, C. Classical Logic and the Strict Tolerant Hierarchy. *Journal of Philosophical Logic 49*, 2 (Apr. 2020), 351–370.

[22] Shapiro, S. Incompleteness and Inconsistency. *Mind 111*, 444 (Oct. 2002), 817–832.