# A Case for Weak Kleene ST*

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**Abstract**

The substructural Strict/Tolerant logic based on strong Kleene valuations ($sST$) was motivated by its ability to express a fully transparent truth predicate and the tolerance principle without falling into the traps of semantic and soritical paradoxes. Even though $sST$ rejects the meta-inferential rule of Cut, it has been shown that many instances of Cut are recoverable. Thus, not only can theories of truth and vagueness based on $sST$ avoid the semantic and soritical paradoxes, but these theories stay very close to classical theories, which is counted as a virtue of $sST$. In a recent paper by Murzi and Rossi, the authors argue that the notion of (un)paradoxicality plays a major role in recapturing the "safe" instances of Cut. However, the theory of truth based on $sST$ cannot be extended to express the notion (un)paradoxicality on pain of revenge paradox. Similarly, in a recent paper by Bruni and Rossi, the authors argue that the theory of vagueness based on $sST$ cannot be extended to express the notion of determinateness on pain of revenge paradox, even though "determinateness" plays a major role in the theory.

In this paper, we argue that given the analysis of these revenge paradoxes, the Strict/Tolerant logician should prefer the weak Kleene variation of the Strict/Tolerant logic ($wST$). We argue that $wST$ can express a fully transparent truth predicate and the tolerance principle as well as the notions of (un)paradoxicality and determinateness (though we prefer to use the notion of groundedness to encompass both of these notions) while still being immune to revenge. We conclude that the logic $wST$ is more appealing than $sST$, for it has the same virtues as $sST$ while it has an unmatched expressive power.

***Keywords***— Strict/Tolerant Logic, Weak Kleene, Paradox, Revenge, Sorites.

The strict/tolerant logic based on strong Kleene valuations ($sST$)[1] has been boasted for its ability to be extended with a fully transparent truth predicate

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[1] $sST$ is often simply called $ST$ in the literature. In fact, sometimes $ST$ stands for the

while still being able to handle semantic paradoxes [4, 5, 17, 18]. It has also been boasted for its ability to be extended with vague predicates as well as the tolerance principle while still being able to handle soritical paradoxes [3, 5]. However, Murzi and Rossi [15] showed that once the theory introduces the notion of paradoxicality, a revenge liar emerges. Similarly, Bruni and Rossi [2] showed that a revenge soritical paradox emerges once the notion of determinateness is introduced.

On the other hand, the weak Kleene flavor of the strict/tolerant logic ($wST$) has been motivated by its ability to handle meaningless or nonsensical sentences and how it fares better than other significance logics [19]. It has also been motivated for, unlike $sST$, its ability to capture content-theoretic defects in bounds consequence interpretation of sequents (i.e., extra-veridical bounds on discourse) such as blasphemies, profanities, and obscenities in [9], and secrecy cases in [8]. As far as we know, these are the papers that were the first to advocate for $wST$.[2]

Our aim in this paper is to show that not only does weak-Kleene-based $ST$ logic avoid the semantic and soritical revenge paradoxes, but also, an analysis of how these paradoxes are constructed in theories based on $sST$ should motivate the Strict/Tolerant logician to adopt weak Kleene valuations.

## 1 Preliminaries

We start with a base language $\mathcal{L}$ which is a standard first-order language, plus the constants $\top$ and $\bot$. The logic $sST$ is the strict/tolerant logic based on strong Kleene models. A strong Kleene (**SK**) model is defined via the following tables ([13], p.334):

| $\neg$ | |
|---|---|
| **0** | 1 |
| **n** | n |
| **1** | 0 |

| $\wedge$ | **0** | **n** | **1** |
|---|---|---|---|
| **0** | 0 | 0 | 0 |
| **n** | 0 | n | n |
| **1** | 0 | n | 1 |

| $\vee$ | **0** | **n** | **1** |
|---|---|---|---|
| **0** | 0 | n | 1 |
| **n** | n | n | 1 |
| **1** | 1 | 1 | 1 |

| $\rightarrow$ | **0** | **n** | **1** |
|---|---|---|---|
| **0** | 1 | 1 | 1 |
| **n** | n | n | 1 |
| **1** | 0 | n | 1 |

plus, $v(\forall x\varphi(x)) = \inf(\{v(\varphi(t) \in \{0, n, 1\} | \, t \text{ is a closed term}\})$ where $n$ is treated as $1/2$ and the order relation is the less-than-or-equal relation between rationals, $v(\top) = 1$, and $v(\bot) = 0$.

On the other hand, the logic $wST$ is the strict/tolerant logic based on weak Kleene models. A weak Kleene (**WK**) model is defined via the following tables (see, [1, 12, 13]):

theory of truth based on strong Kleene $ST$ logic. To avoid any confusion, we will keep calling the strict/tolerant logic based on the strong Kleene valuations $sST$ while we will call the theory of truth based on $sST$, $sSTT$.

[2]However, there is an earlier appearance of a $wST$-like system. That system is briefly discussed in ([6], §7). Many thanks to the anonymous reviewer for pointing this out.

| $\neg$ | |
|---|---|
| **0** | 1 |
| **n** | n |
| **1** | 0 |

| $\wedge$ | **0** | **n** | **1** |
|---|---|---|---|
| **0** | 0 | n | 0 |
| **n** | n | n | n |
| **1** | 0 | n | 1 |

| $\vee$ | **0** | **n** | **1** |
|---|---|---|---|
| **0** | 0 | n | 1 |
| **n** | n | n | n |
| **1** | 1 | n | 1 |

| $\rightarrow$ | **0** | **n** | **1** |
|---|---|---|---|
| **0** | 1 | n | 1 |
| **n** | n | n | n |
| **1** | 0 | n | 1 |

plus, $v(\forall x\varphi(x)) = \inf(v(\varphi(t)) \mid t$ is a closed term$\}$ if every $\varphi(t) \in \{0,1\}$, otherwise $v(\forall x\varphi(x)) = n$, and similar to the **SK**-model, $v(\top) = 1$ and $v(\bot) = 0$.[3]

Validity is defined via counterexamples as follows (see, [18] p.150):
An $sST$-counterexample to $\Gamma \vDash \Delta$ is an **SK**-model such that $\forall\gamma \in \Gamma, v(\gamma) = 1$ and $\forall\delta \in \Delta, v(\delta) = 0$. A sequent is $sST$-valid iff no **SK** model is an $sST$-counterexample to it.

Similarly, A $wST$-counterexample to $\Gamma \vDash \Delta$ is a **WK**-model such that $\forall\gamma \in \Gamma, v(\gamma) = 1$ and $\forall\delta \in \Delta, v(\delta) = 0$. A sequent is $wST$-valid iff no **WK** model is a $wST$-counterexample to it.

One preliminary note is that we will keep transitioning between proof theory and model theory throughout the paper. For soundness and completeness results for $sST$, the reader can refer to [18] and [7], and for soundness and completeness results for $wST$, the reader can refer to [16] and [9].

Moreover, Murzi and Rossi rely on the notion of *unparadoxicality* to construct their semantic revenge argument, while Bruni and Rossi use the notion of *determinateness* to construct their soritical revenge argument. For our purposes, we will use the notion of *groundedness* to encompass both of these notions. Of course, the notion of groundedness is not synonymous with unparadoxicality. For instance, the truth-teller is ungrounded but not paradoxical. However, none of the arguments below hinge on our choice of using the notion of groundedness instead of unparadoxicality and determinateness. As we will see, Murzi and Rossi's rules for the unparadoxicality predicate can naturally be expressed using a groundedness predicate. The same applies to the determinateness predicate that is used in Bruni and Rossi's work. If the reader is uncomfortable with the notion of groundedness, they can replace the groundedness predicate with an unparadoxicality predicate in section 2 and with a determinateness predicate in section 3 without any detraction from the arguments.[4]

## 2 An Argument from Semantic Revenge

In [15], Julien Murzi and Lorenzo Rossi present a revenge problem to a theory of truth and paradox built on $sST$. Murzi and Rossi ([15], p.166) employ the notion of unparadoxicality, but for our purposes, we will employ the notion of groundedness. A sentence is *grounded* if and only if its value is $\in$ {0,1}, otherwise, it is ungrounded.

To construct Murzi and Rossi's revenge paradox, we will expand our base language to include a transparent truth predicate and a groundedness predicate.

[3] Technically speaking, we take $\wedge$, $\neg$, and $\forall$ to be primitive connectives, and $\vee$, $\rightarrow$, and $\exists$ are defined via these primitive connective as usual.

[4] My gratitude to the anonymous reviewer for urging me to emphasize this point.

Both the **SK**-models and the **WK**-models are expanded to interpret these two new predicates. Where "$\ulcorner\urcorner$" is a naming device:

- $v(Tr(\ulcorner\varphi\urcorner)) = v(\varphi)$.
- $v(Grd(\ulcorner\varphi\urcorner)) = 1$ *iff* $v(\varphi) \in \{0, 1\}$ and $v(Grd(\ulcorner\varphi\urcorner)) = 0$ *iff* $v(\varphi) = n$.[5]

We will call the theory of truth and groundedness $sSTTG$ if it is based on strong Kleene models and $wSTTG$ if it is based on weak Kleene models.

The rules for the groundedness predicate follow Murzi and Rossi's rules for unparadoxicality:[6]

$$\dfrac{\begin{array}{c} \square\ \overline{\Gamma\vdash\varphi}^{\,n} \qquad \square\ \overline{\Delta,\varphi\vdash\psi}^{\,n} \\ \ddots \qquad \iddots \\ \vdots \\ \Gamma,\Delta\vdash\psi \end{array}}{\Gamma,\Delta\vdash Grd(\ulcorner\varphi\urcorner)}\ Grd\text{-I}, n$$

$$\dfrac{\Gamma\vdash Grd(\ulcorner\varphi\urcorner) \qquad \Delta_0\vdash\varphi \qquad \Delta_1,\varphi\vdash\psi}{\Gamma,\Delta_0,\Delta_1\vdash\psi}\ \text{Grd-E}$$

Thus, the rule $Grd$-I says that if $\varphi$ is cuttable, then $\varphi$ is grounded. On the other hand, $Grd$-E says if a sentence $\varphi$ is grounded, then Cut can be performed on $\varphi$ ([15], p.166). Using the rules stated in [15], we can construct Murzi and Rossi's revenge argument. Let $\zeta$ be $\neg Tr(\ulcorner\zeta\urcorner) \wedge Grd(\ulcorner\zeta\urcorner)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\zeta\vdash\zeta}\ \text{id} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\zeta\vdash\zeta}\ \text{id}}{\zeta\vdash\neg Tr(\ulcorner\zeta\urcorner)\wedge Grd(\ulcorner\zeta\urcorner)}\ \text{def of }\zeta}{\zeta\vdash\neg Tr(\ulcorner\zeta\urcorner)}\ \wedge\text{-E}_1}{\zeta\vdash\neg\zeta}\ \neg\text{-Tr-E}}{\zeta,\zeta,\zeta\vdash\bot}\ \neg\text{-E}_W\,c}{\zeta,\zeta\vdash\bot}\ \text{Contraction}}{\zeta\vdash\bot}\ \text{Contraction}$$

Call this derivation $D_0$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\zeta}\,1}{\vdash\neg Tr(\ulcorner\zeta\urcorner)\wedge Grd(\ulcorner\zeta\urcorner)}\ \text{Def of }\zeta}{\vdash Grd(\ulcorner\zeta\urcorner)}\ \wedge\text{-E}_2 \qquad \dfrac{}{\vdash\zeta}\,1 \qquad \dfrac{}{\zeta\vdash\bot}\,1}{\vdash\bot}\ Grd\text{-E}}{\vdash Grd(\ulcorner\zeta\urcorner)}\ Grd\text{-I}, 1$$

[5] Thus, $Grd$ is a bivalent predicate.

[6] Murzi and Rossi are using Natural Deduction in Sequent-calculus style. Moreover, in $Grd - I$, $\psi$ should be carried to the conclusion of the last sequent, but since $\psi$ will be instantiated with $\bot$ in our proofs, it would not matter since $\bot$ is cuttable given $\bot\vdash$.

Call this derivation $D_1$.

$$
\dfrac{\begin{array}{c} D_1 \\ \vdash Grd(\ulcorner\zeta\urcorner) \end{array} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} D_0 \\ \zeta \vdash \bot \end{array}}{\vdash \neg\zeta}\;\neg\text{-I}}{\vdash \neg Tr(\ulcorner\zeta\urcorner)}\;\neg\text{-Tr-I} \qquad \begin{array}{c} D_1 \\ \vdash Grd(\ulcorner\zeta\urcorner) \end{array}}{\vdash \neg Tr(\ulcorner\zeta\urcorner) \wedge Grd(\ulcorner\zeta\urcorner)}\;\wedge\text{-I}}{\vdash \zeta}\;\text{def of }\zeta \qquad \begin{array}{c} D_0 \\ \zeta \vdash \bot \end{array}}{\vdash \bot}\; Grd\text{-E}
$$

We claim that conjunction elimination is the suspect for the aforementioned revenge argument. A weak-Kleene-based system, however, avoids the revenge problem because it takes ungroundedness to be infectious. So, we will now address why conjunction elimination is the problem and why the weak-Kleene-based *ST* is the correct solution for the *ST*-theorist.

The revenge argument that targets *sSTTG* relies on two claims:

1. For any $\varphi$, $\varphi$ is cuttable if and only if $\varphi$ is grounded.
2. If a formula $\varphi$ is a conjunction, $\mathcal{A} \wedge \mathcal{B}$, then the following instance of Cut

$$
\dfrac{\Gamma \vdash \mathcal{A} \wedge \mathcal{B}, \Delta \qquad \Gamma, \mathcal{A} \wedge \mathcal{B} \vdash \mathcal{A}, \Delta}{\Gamma \vdash \mathcal{A}, \Delta}
$$

   holds in *sSTT*, and similarly for the other conjunct, $\mathcal{B}$.

Claim 1 follows directly from the *Grd* rules. Claim 2 follows from the fact that $\Gamma \vDash_{sST} \mathcal{A}, \Delta$ follows from $\Gamma \vDash_{sST} \mathcal{A} \wedge \mathcal{B}, \Delta$. To see this, suppose $\Gamma \nvDash_{sST} \mathcal{A}, \Delta$. Thus, for some **SK**-model, $\forall\gamma \in \Gamma$, $v(\gamma) = 1$, $\forall\delta \in \Delta$, $v(\delta) = 0$, and $v(\mathcal{A}) = 0$. Thus, $v(\mathcal{A} \wedge \mathcal{B})$ must equal 0. Therefore, $\Gamma \nvDash_{sST} \mathcal{A} \wedge \mathcal{B}, \Delta$. Thus, given completeness, it follows that the following instance of Cut holds:

$$
\dfrac{\dfrac{\vdots}{\Gamma \vdash \mathcal{A} \wedge \mathcal{B}, \Delta} \qquad \dfrac{\Gamma, \mathcal{A} \vdash \mathcal{A}, \Delta}{\Gamma, \mathcal{A} \wedge \mathcal{B} \vdash \mathcal{A}, \Delta}\;\wedge\vdash}{\Gamma \vdash \mathcal{A}, \Delta}\;\text{Cut}
$$

Because $\mathcal{A} \wedge \mathcal{B}$ is cuttable, then given claim 1, $\mathcal{A} \wedge \mathcal{B}$ is grounded as long as there is a derivation of $\Gamma \vdash \mathcal{A} \wedge \mathcal{B}, \Delta$.[7]

Note, that the fact that *Grd* is a bivalent predicate is playing no role here. We could have used $\top$ as the second conjunct. Though, it seems that in Murzi and Rossi's, the conjunct being *Grd* was necessary. However, given how they defined *Grd*-I, we do not need *Grd*-E to get *Grd*-I for $\zeta$. Here is a possible way to modify Murzi and Rossi's derivation $D_1$:

[7] For more information on the completeness of *sST*, refer to Dicher and Paoli's [7]. To have a proof system that is complete with regards to *sST*, the authors claim that the proof system *LK* without Cut must be extended with a few rules. Among these rules are the conjunction elimination rules.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\scriptsize 1}
\UnaryInfC{$\vdash \zeta$}
\RightLabel{\scriptsize Def of $\zeta$}
\UnaryInfC{$\vdash \neg Tr(\ulcorner\zeta\urcorner) \wedge Grd(\ulcorner\zeta\urcorner)$}
\RightLabel{\scriptsize $\wedge$-E$_2$}
\UnaryInfC{$\vdash Grd(\ulcorner\zeta\urcorner)$}
\AxiomC{}
\RightLabel{\scriptsize 1}
\UnaryInfC{$\zeta \vdash \bot$}
\RightLabel{\scriptsize Tr-I}
\UnaryInfC{$Tr(\ulcorner\zeta\urcorner) \vdash \bot$}
\RightLabel{\scriptsize $\neg$-I}
\UnaryInfC{$\vdash \neg Tr(\ulcorner\zeta\urcorner)$}
\RightLabel{\scriptsize $\wedge$-I}
\BinaryInfC{$\vdash Grd(\ulcorner\zeta\urcorner) \wedge \neg Tr(\ulcorner\zeta\urcorner)$}
\RightLabel{\scriptsize $\wedge$-E}
\UnaryInfC{$\vdash Grd(\ulcorner\zeta\urcorner)$}
\RightLabel{\scriptsize Weakening}
\UnaryInfC{$\vdash Grd(\ulcorner\zeta\urcorner), \bot$}
\RightLabel{\scriptsize $Grd$-I, 1}
\UnaryInfC{$\vdash Grd(\ulcorner\zeta\urcorner)$}
\end{prooftree}
$$

This works because our sentence $\zeta$ disappeared and we followed the rule $Grd$-I. You can see it more clearly when you replace the conjunct $Grd(\ulcorner\zeta\urcorner)$ with $\top$. Thus, $Grd$-E is not needed to prove that $\zeta$ is grounded, and revenge is not confined to sentences that are defined with the groundedness predicate. [8]

Moreover, revenge is not confined to conjunctions either. For instance, $\mathcal{A}, \Gamma \vdash \Delta$ is derivable from $\mathcal{A} \vee \mathcal{B}, \Gamma \vdash \Delta$, similarly with conditionals. In other words, we can construct a revenge argument using the sentence $\kappa$ where $\kappa$ is $\neg Tr(\ulcorner\kappa\urcorner) \vee \neg Grd(\ulcorner\kappa\urcorner)$.[9] Thus, the aforementioned claim 2 can be expanded to include disjunctions and conditionals.

Similar to what we said about $\zeta$, the second disjunct in $\kappa$ need not have a $Grd$ predicate; we can have $\bot$ instead. $\kappa$ is equivalent to a Curry sentence, so we can do the same thing for a Curry sentence with a false consequent. That means, once we introduce the $Grd$ rules, Curry sentences become cuttable; trivializing the theory $sSTTG$. As Murzi and Rossi put it:"(among others) the theories developed by Ripley and Cobreros et al. cannot express the notion '$\varphi$ behaves classically given a derivation of $\vdash \psi$ from $\vdash \varphi$ and $\varphi \vdash \psi$', on pain of triviality" ([15], p.167). So, a proponent of $sSTT$ who wants to express the notion of groundedness must reject claim 1 since claim 2 would entail rejecting strong Kleene models. But what does it mean to reject claim 1? We will argue that it equally means that it must reject strong Kleene models if we want our logic to express the notions of admissibility and groundedness.

To reject claim 1 is to either reject the claim that if $\varphi$ is cuttable, then $\varphi$ is grounded, or (inclusive) reject the claim that if $\varphi$ is grounded, then $\varphi$ is cuttable. In other words, we either reject $Grd$-I, reject $Grd$-E, or reject both of these rules. Suppose we reject $Grd$-I. This entails that (some) ungrounded sentences are cuttable. This is in line with the fact that conjunction elimination is derivable regardless of whether the conjunction is grounded or ungrounded. The question remains how are we to demarcate between grounded and ungrounded sentences proof theoretically in $sSTTG$? It is not clear that this is possible. However, even if it is possible, this may avoid the paradox using the sentence $\sigma$ defined as $\neg Tr(\ulcorner\sigma\urcorner) \wedge \top$ by claiming that $\sigma$ is actually ungrounded, but it would not relieve the paradox using Murzi and Rossi's sentence, $\zeta$, because $\zeta$ must be grounded.

[8]To ensure that the bivalence of $Grd$ plays a role, maybe we want to insist that the proof leading to $Grd$-I must be normal (i.e., avoid the detour in the steps $\wedge$-I and $\wedge$-E). However, discussing this issue further would take us beyond the scope of the paper.

[9]This is not surprising since we define $\vee$ and $\rightarrow$ in terms of $\neg$ and $\wedge$.

To see this, let us analyze the sentences $\sigma$ and $\zeta$ semantically. If $v(\sigma) = v(\neg Tr(\ulcorner\sigma\urcorner) \wedge \top) = 1$, then $v(Tr(\ulcorner\sigma\urcorner))$ must also equal 1. As a result, the first conjunct must be false, rendering $v(\sigma) = 0$ $*$contradiction$*$. Similarly, if $v(\sigma) = v(\neg Tr(\ulcorner\sigma\urcorner) \wedge \top) = 0$, then $v(Tr(\ulcorner\sigma\urcorner))$ must equal 0 and the first conjunct must be true. Rendering $v(\sigma) = 1$ $*$contradiction$*$. However, if $v(\sigma) = v(\neg Tr(\ulcorner\sigma\urcorner) \wedge \top) = n$, then there would be no contradiction. So, $v(\sigma)$ must be $n$.

This line of reasoning is not available for $\zeta$ given the bivalence of the groundedness predicate. That is, every sentence is either grounded or not. If $v(\zeta) = v(\neg Tr(\ulcorner\zeta\urcorner) \wedge Grd(\ulcorner\zeta\urcorner)) = n$, then $\zeta$ is ungrounded. Hence, the second conjunct is false. Under strong Kleene valuations, $\zeta$ must be false $*$contradiction$*$. So, if $v(\zeta) \neq n$, then $\zeta$ must be grounded and the second conjunct is true. Thus, if $\zeta$ is either true or false, a contradiction would follow given the value of the first conjunct must be the opposite of the value of $\zeta$. [10]

What about the other direction of claim 1? Suppose we reject *Grd*-E. This means that not all grounded Cut-formulas are cuttable. So, the strict/tolerant logician must claim that there are instances where $\Gamma \vDash \varphi, \Delta$ and $\Sigma, \varphi \vDash \Xi$, but $\Gamma, \Sigma \nvDash \Delta, \Xi$ where $\varphi$ is a grounded sentence. Given $\Gamma, \Sigma \nvDash \Delta, \Xi$, it follows that every sentence in $\Gamma$ and $\Sigma$ receives the value 1 and every sentence in $\Delta$ and $\Xi$ receives the value 0. Given that $\varphi$ is grounded, $v(\varphi)$ is either 1 or 0. If $v(\varphi) = 1$ then $\Sigma, \varphi \nvDash \Xi$. If $v(\varphi) = 0$, then $\Gamma \nvDash \varphi, \Delta$. Thus, it is incoherent for the strict/tolerant logician to reject *Grd*-E.

To prevent Murzi and Rossi's paradox, we must treat $\zeta$ as an ungrounded sentence. This pushes us to treat an ungrounded conjunct as infectious. That is, a conjunction of an ungrounded conjunct and a false conjunct must be ungrounded. In other words, conjunction must obey weak Kleene valuations. In a theory based on weak Kleene valuations, conjunction elimination does not hold. Here is a counterexample: let $\lambda$ be an ungrounded sentence, and suppose $\vDash_{wSTT} \lambda \wedge \bot$, $\nvDash_{wSTT} \bot$. So Murzi and Rossi's revenge argument would not go through in $wSTTG$. However, conjunction elimination for conjunctions that receive the value 1 or 0 can easily be recovered. That is, we can recapture conjunction elimination on grounded conjunctions.

A natural question is whether there can be other revenge arguments in $wSTTG$. In [11], Anil Gupta and Robert Martin show that, in general, a language based on weak Kleene valuations *can* contain its truth predicate and also its "ungroundedness" predicate (see also, ([10], p.80-83)).[11] However, one might wonder how the general result in Gupta and Martin's [11] applies to $wSTTG$.[12]

Gupta and Martin's proof relies on two properties (actually, there is a third unstated assumption, but mentioned in [10], p.81—that the grounded-

[10] One might argue that $\zeta$ is neither grounded nor ungrounded; its groundedness is indeterminate. This would only push the problem further, and another revenge argument can be constructed using this new notion of indeterminacy.

[11] Another appeal to Gupta and Martin's result as a selling point for $wST$ can be found in ([9], fn.6). Ferguson utilizes their result to show that a "repugnance" predicate can be expressed in a theory based on $wST$.

[12] Many thanks to the anonymous reviewer for raising this question.

ness/ungroundedness predicates are bivalent): The first property, like in Kripke's [14], is that as the extension and anti-extensions increase, whatever is true, stays true and whatever is false, stays false. The second property is that the sentences that are ungrounded are ungrounded in all models that agree on the domain and agree on all of the predicates' ranges of applications in the language (for formal details, see [11], p.132).

What is important to note here is that these properties are not properties of consequence relations but properties of valuation schemes. The weak Kleene valuation schema fulfills these properties, whereas the strong Kleene schema does not, as is evident by strengthened liar sentences. In other words, just as revenge arguments do not only inflict $sSTTG$ but all theories that adopt the strong Kleene schema (as it is shown in [15] that theories based on $LP$ and $K_3$ are also subject to revenge arguments), all theories based on the weak Kleene schema can express that third value as a bivalent predicate without the risk of revenge arguments. That is, theories based on weak Kleene variants of $LP$ and $K_3$ would also block the revenge arguments in [15]. For instance, in the proofs provided by Murzi and Rossi, the conjunction elimination step would not be allowed in weak Kleene $LP$ whereas the disjunction introduction step would not be permitted in weak Kleene $K_3$. However, since we are working with $wSTTG$ in particular, we want to show that $wSTTG$ is revenge-free.

**Theorem 1.** The theory $wSTTG$ is revenge-free.

*Proof.* Assume that there is a revenge argument in $wSTTG$. That means that there is a sentence $\varrho$ such that $\vDash Grd(\ulcorner\varrho\urcorner)$, $\vDash \varrho$, and $\varrho \vDash \bot$. Put $\varrho$ in its conjunctive normal form. Since $\varrho$ is a revenge sentence, one of its conjunct uses $Grd$ predicate (i.e., one of the conjuncts is $Grd(\ulcorner\varrho\urcorner)$ or $\neg Grd(\ulcorner\varrho\urcorner)$). Because $Grd$ is bivalent and $\vDash Grd(\ulcorner\varrho\urcorner)$, $v(Grd(\ulcorner\varrho\urcorner)) = 1$.

Now, either $\varrho := ... \wedge Grd(\ulcorner\varrho\urcorner)$ or $\varrho := ... \wedge \neg Grd(\ulcorner\varrho\urcorner)$. If none of the conjuncts receive the value $n$, then $v(\varrho) = 1$ or $v(\varrho) = 0$. If $(v(\varrho) = 1)$, then $\varrho \nvDash \bot$, *contradiction*. If $(v(\varrho) = 0)$, then $\nvDash \varrho$, *contradiction*.

If at least one of the conjuncts receives the value $n$, then $v(\varrho) = n$ given weak Kleene valuations. Thus, $v(Grd(\ulcorner\varrho\urcorner)) = 0$ and so $\nvDash Grd(\ulcorner\varrho\urcorner)$, *contradiction*. Therefore, there can be no revenge argument in $wSTTG$. □

One might argue that even if $wSTTG$ avoids revenge paradoxes, there are still semantic concepts that play some explanatory role in the theory yet cannot be represented in the object language. For instance, the notion of "grounded truth" plays a role in our theory, but if we add this concept to the object language, we would face revenge paradoxes. To see this, let $v(GrdTr(\ulcorner\varphi\urcorner) = 1$ if $v(\varphi) = 1$ and 0 otherwise. We would reach a revenge paradox with the liar-like sentence $\pi$ such that $\pi$ is equivalent to $\neg GrdTr(\ulcorner\pi\urcorner)$. Suppose that the $v(\pi) = 1$. It follows that the $v(GrdTr(\ulcorner\pi\urcorner)) = 1$, and thus, $v(\neg GrdTr(\ulcorner\pi\urcorner)) = 0$. As a result, $v(\pi) = 0$; *contradiction*. Suppose $v(\pi) = 0$. So, $v(\neg GrdTr(\ulcorner\pi\urcorner)) = 1$, and hence, $v(\pi) = 1$; *contradiction*. Finally, suppose $v(\pi) = n$. Thus, $v(GrdTr(\ulcorner\pi\urcorner)) = 0$ and $v(\neg GrdTr(\ulcorner\pi\urcorner)) = 1$. So, $v(\pi) = 1$; *contradiction*. Therefore, there are no weak Kleene models satisfying the semantics for $GrdTr$.

It is true that the theory $wSTTG$ utilizes the notion of "grounded truth". However, when we add semantic notions to our language, the semantic notion must expand our expressive power. "Grounded truth", even if it does not lead to triviality, does not expand the expressive power of $wSTTG$. That is because we can already express "grounded truth" in $wSTTG$. If we want to say that $\varphi$ is a grounded truth, we simply say that $\varphi$ is grounded and true, which is expressible in $wSTTG$: $Grd(\ulcorner\varphi\urcorner) \wedge Tr(\ulcorner\varphi\urcorner)$. As we have argued above, such sentences would not lead to triviality in $wSTTG$ even if they were self-referential. Thus, to show that there are revenge paradoxes on theories that expand $wSTTG$, we must show not only that there are semantic notions that play a role in $wSTTG$ that cannot be accommodated but also that we *need* to expand $wSTTG$ to express such notions.

## 3 An Argument from Soritical Revenge

We can also construct a soritical revenge argument against $sSTVG$ (where $V$ stands for vagueness). As noted by Bruni and Rossi [2], a revenge liar and a soritical revenge paradox share some common features. We will set up the revenge soritical argument in a slightly different way from Bruni and Rossi's [2], but the two arguments have the same force against the expanded $sSTV$. Bruni and Rossi use a bivalent determinateness operator, while we will keep using our bivalent groundedness predicate. We expand our base language with a vague predicate $P$ and a two-place relation $\sim_P$, as well as a groundedness predicate as defined earlier. Let $P$ be your favorite vague predicate (e.g., "is tall", "is bald",...,etc.), and $c_i \sim_P c_{i+1}$ to mean that $c_i$ is indistinguishable from $c_{i+1}$ with respect to the predicate $P$. To set up the soritical revenge argument, we will assume, like Bruni and Rossi ([2], p.12-13), the following:

1. $v(P(c_0)) = 1$
2. There is an object $c_j$ where $v(P(c_j)) = n$
3. There is an object $c_r$ where $v(P(c_r)) = 0$
4. For every $i$, $v(c_i \sim_P c_{i+1}) = 1$
5. $v(P(c_e)) \geq v(P(c_f))$ *iff* e $\leq$ f when we take the order of the truth values as $< 0, n, 1 >$.

Finally, we will assume:

**(Tolerance-G)** $\quad P(c_i) \wedge Grd(\ulcorner P(c_i)\urcorner) \wedge c_i \sim_P c_j \vdash P(c_j) \wedge Grd(\ulcorner P(c_j)\urcorner)$

This tolerance principle follows instantly from the usual tolerance principle used in soritical arguments. The only difference is that we are focusing on the grounded sentences. In other words, if $P(c_i)$ is a true grounded sentence and $c_i$ is indistinguishable from $c_j$ with respect to $P$, then $P(c_j)$ is a true and grounded sentence as well.

From the assumption 1−5, there is a smallest $i$ such that $v(P(c_i)) = n$. Let $r$ be that smallest $i$. In other words, $P(c_r)$ is the first sentence that receives the value $n$, and hence, $P(c_r)$ is ungrounded. Let $P(c_q)$ be the last sentence to receive the value 1. Hence, we know that $P(c_q)$ is true and grounded, and $c_q$ is indistinguishable from $c_r$ with respect to $P$. It follows that $sSTVG$ is trivial:

$$\dfrac{\vdash P(c_q) \wedge Grd(\ulcorner P(c_q)\urcorner) \wedge c_q \sim_P c_r \qquad P(c_q) \wedge Grd(\ulcorner P(c_q)\urcorner) \wedge c_q \sim_P c_r \vdash P(c_r) \wedge Grd(\ulcorner P(c_r)\urcorner)}{\vdash P(c_r) \wedge Grd(\ulcorner P(c_r)\urcorner)}\ \text{Cut}$$

Call this derivation $D_0$

$$\dfrac{\dfrac{D_0}{\vdash P(c_r) \wedge Grd(\ulcorner P(c_r)\urcorner)} \qquad \dfrac{Grd(\ulcorner P(c_r)\urcorner) \vdash}{P(c_r) \wedge Grd(\ulcorner P(c_r)\urcorner) \vdash}\ \wedge\vdash}{\vdash}\ \text{Cut}$$

All that is left to show is that these instances of Cut do indeed hold in $sSTVG$.[13]

**Proposition 1.** $P(c_k) \wedge Grd(\ulcorner P(c_k)\urcorner) \wedge c_k \sim_P c_{k+1}$ is cuttable in $sSTVG$. That is, if $\Gamma \vDash_{sSTVG} \Delta, P(c_k) \wedge Grd(\ulcorner P(c_k)\urcorner) \wedge c_k \sim_P c_{k+1}$ and $\Gamma, P(c_k) \wedge Grd(\ulcorner P(c_k)\urcorner) \wedge c_k \sim_P c_{k+1} \vDash_{sSTVG} \Delta$, then $\Gamma \vDash_{sSTVG} \Delta$.

*Proof.* Suppose $\Gamma \nvDash_{sSTVG} \Delta$. So, $\forall\gamma \in \Gamma, v(\gamma) = 1$ and $\forall\delta \in \Delta, v(\delta) = 0$. Since groundedness is bivalent, $v(Grd(\ulcorner P(c_k)\urcorner))$ is either 0 or 1. If $v(Grd(\ulcorner P(c_k)\urcorner)) = 0$, then given strong Kleene valuation, $v(P(c_k) \wedge Grd(\ulcorner P(c_k)\urcorner) \wedge c_k \sim_P c_{k+1}) = 0$, and hence, $\Gamma \nvDash_{sSTVG} \Delta, P(c_k) \wedge Grd(\ulcorner P(c_k)\urcorner) \wedge c_k \sim_P c_{k+1}$.

If $v(Grd(\ulcorner P(c_k)\urcorner)) = 1$, then $v(P(c_k))$ is either 1 or 0. If $v(P(c_k)) = 0$, then given strong Kleene valuation, $v(P(c_k) \wedge Grd(\ulcorner P(c_k)\urcorner) \wedge c_k \sim_P c_{k+1}) = 0$, and hence, $\Gamma \nvDash_{sSTVG} \Delta, P(c_k) \wedge Grd(\ulcorner P(c_k)\urcorner) \wedge c_k \sim_P c_{k+1}$. If $v(P(c_k)) = 1$, then since $v(c_k \sim_P c_{k+1}) = 1$ by assumption, $v(P(c_k) \wedge Grd(\ulcorner P(c_k)\urcorner) \wedge c_k \sim_P c_{k+1}) = 1$. Thus, $\Gamma, P(c_k) \wedge Grd(\ulcorner P(c_k)\urcorner) \wedge c_k \sim_P c_{k+1} \nvDash_{sSTVG} \Delta$. □

**Proposition 2.** $P(c_i) \wedge Grd(\ulcorner P(c_i)\urcorner)$ is cuttable in $sSTVG$. That is, if $\Gamma \vDash_{sSTVG} \Delta, P(c_i) \wedge Grd(\ulcorner P(c_i)\urcorner)$ and $\Gamma, P(c_i) \wedge Grd(\ulcorner P(c_i)\urcorner) \vDash_{sSTVG} \Delta$, then $\Gamma \vDash_{sSTVG} \Delta$.

*Proof.* Suppose $\Gamma \nvDash_{sSTVG} \Delta$. So, $\forall\gamma \in \Gamma, v(\gamma) = 1$ and $\forall\delta \in \Delta, v(\delta) = 0$. Since groundedness is bivalent, $v(Grd(\ulcorner P(c_i)\urcorner))$ is either 0 or 1. If $v(Grd(\ulcorner P(c_i)\urcorner)) = 0$, then $v(P(c_i) \wedge Grd(\ulcorner P(c_i)\urcorner)) = 0$, and hence $\Gamma \nvDash_{sSTVG} \Delta, P(c_i) \wedge Grd(\ulcorner P(c_i)\urcorner)$. If $v(Grd(\ulcorner P(c_i)\urcorner)) = 1$, then $v(P(c_i))$ is either 0 or 1. If $v(P(c_i)) = 0$, then $v(P(c_i) \wedge Grd(\ulcorner P(c_i)\urcorner)) = 0$, and hence $\Gamma \nvDash_{sSTVG} \Delta, P(c_i) \wedge Grd(\ulcorner P(c_i)\urcorner)$. If $v(P(c_i)) = 1$ then $v(P(c_i) \wedge Grd(\ulcorner P(c_i)\urcorner)) = 1$, and hence, $\Gamma, P(c_i) \wedge Grd(\ulcorner P(c_i)\urcorner) \nvDash_{sSTVG} \Delta$ □

As observed by Bruni and Rossi [2], just as $sSTT$ cannot be expanded to express the notion of unparadoxicality (groundedness) on pain of paradox, $sSTV$

[13] The proofs of the following four propositions are done semantically. Hence, in what follows, we are assuming soundness and completeness.

cannot be expanded to express the notion of determinateness (groundedness) on pain of paradox.

If we take the assumptions 1−5 + Tolerance-G to be fair assumptions and we want our ST logic to express the notion of groundedness, then the only way out is to reject the admissibility of Cut in the aforementioned instances. To reject the admissibility of Cut in these instances is to reject the strong Kleene valuation. Thus, an ST-logic based on weak Kleene valuation is the most attractive option for a proponent of ST. After all, the aforementioned instances of Cut, in their general forms, do not hold in $wSTVG$.

**Proposition 3.** *$P(c_k) \wedge Grd(\ulcorner P(c_k) \urcorner) \wedge c_k \sim_P c_{k+1}$ is not cuttable in $wSTVG$.*

*Proof.* $v(c_k \sim_P c_{k+1}) = 1$ by assumption. Let $v(P(c_k)) = n$ and $v(Grd(\ulcorner P(c_k) \urcorner)) = 0$. By weak Kleene valuation, $v(P(c_k) \wedge Grd(\ulcorner P(c_k) \urcorner) \wedge c_k \sim_P c_{k+1}) = n$. Thus $\top \vDash_{wSTVG} \bot, P(c_k) \wedge Grd(\ulcorner P(c_k) \urcorner) \wedge c_k \sim_P c_{k+1}$ and $\top, P(c_k) \wedge Grd(\ulcorner P(c_k) \urcorner) \wedge c_k \sim_P c_{k+1} \vDash_{wSTVG} \bot$, but $\top \nvDash_{wSTVG} \bot$. □

**Proposition 4.** *$P(c_i) \wedge Grd(\ulcorner P(c_i) \urcorner)$ is not cuttable in $wSTVG$.*

*Proof.* Let $v(P(c_i)) = n$ and $v(Grd(\ulcorner P(c_i) \urcorner)) = 0$. Thus, by weak Kleene valuation, $v(P(c_i) \wedge Grd(\ulcorner P(c_i) \urcorner)) = n$. It follows that $\top \vDash_{wSTVG} \bot, P(c_i) \wedge Grd(\ulcorner P(c_i) \urcorner)$ and $\top, P(c_i) \wedge Grd(\ulcorner P(c_i) \urcorner) \vDash_{wSTVG} \bot$, but $\top \nvDash_{wSTVG} \bot$. □

One thing to note here is that since the conjuncts $P(c_q)$, $Grd(P(c_q))$, and $c_q \sim_P c_r$ are all grounded, the first instance of Cut in the revenge argument is recoverable. In other words, even though Cut does not hold on $P(c_k) \wedge Grd(\ulcorner P(c_k) \urcorner) \wedge c_k \sim_P c_{k+1}$ in $wSTVG$ in its full generality, it holds in instances where each conjunct is grounded. Thus, the first instance of Cut in the revenge argument can still go through in $wSTVG$. However, the second instance of Cut still does not hold for it has an ungrounded conjunct, and the revenge argument would be blocked.

Weak-Kleene-based ST has the same virtues as strong-Kleene-based ST; it supports a transparent truth, the principle of tolerance, and stays very close to classical logic. However, unlike strong-Kleene-based ST, it avoids the semantic and soritical revenge paradoxes while still being able to express the notion of groundedness (or determinateness and unparadoxicality). Just as how non-classical logics were motivated on the grounds of their abilities to express a transparent notion of truth and to validate the principle of tolerance, we are motivating $wST$ on the grounds of its ability to express the notion of groundedness which plays an essential role in classical recapturing. As Murzi and Rossi ([15], p.168) put it,

> [J]ust as there are strong reasons for wanting truth to be naïve, and hence to adopt one of the non-classical logics [...], there are parallel reasons for wanting paradoxical and unparadoxicality to also be naïve, and hence to adopt an even weaker non-classical logic—one in which the [revenge arguments] no longer go through.

This is exactly what we are motivating. Though, instead of a notion of unparadoxicality, we preferred a notion of groundedness. However, $wST$ is by no means weak. Just as how $sSTT$ can recapture $\rightarrow$-E, $\neg$-E, and $\vee$-E (see [15], p.161), $wSTT$ can recapture these rules as well as $\wedge$-E. Not only can it recapture these rules, but it can also express its recapturing abilities, unlike the strong Kleene version.

## 4 Conclusion

We analyzed Murzi and Rossi's revenge argument against a theory based on $sST$. Our analysis pointed out that the culprit of this revenge argument is conjunction elimination. We argued that the conjunction in this revenge argument should be taken as an ungrounded sentence. Given strong Kleene valuations, it is impossible to count that conjunction as ungrounded, because that would render the second conjunct to be false and would result in a false and grounded conjunction. Weak Kleene valuations would be able to take the conjunction to be ungrounded even if one of the conjuncts is false. A theory based on weak Kleene ST would block the revenge argument since conjunction elimination does not hold in the theory. Nevertheless, the theory can still classically-recapture conjunction elimination.

We, then, analyzed Bruni and Rossi's soritical revenge argument, and we saw the same issue reappears here. We showed how a theory of vagueness based on $wST$ can express the notion of groundedness while it still blocks the soritical revenge argument. Moreover, this theory still validates the commonsensical assumptions that we have made about vague predicates along with the tolerance principle.

Given how theories based on $wST$ can avoid revenge paradoxes while still being extremely close to classical logic and given their expressive power (i.e., being able to non-trivially accommodate crucial semantic notions), the $ST$ theorist should be enticed to adopt weak Kleene $ST$ over strong Kleene $ST$.[14]

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