Classical Logic of Paradox*

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Abstract

We begin with the claim that a paradoxical sentence is a sentence that cannot consistently be true and cannot consistently be false either. We provide two approaches to interpret that claim: a gappy approach that takes paradoxical sentences to be neither true nor false, and a glutty approach that takes paradoxical sentences to be both true and false. We present two systems capable of expressing this semantic understanding. The models that we use ensure that paradoxicality is understood as a bivalent notion—saying of a sentence that it is paradoxical will result in either a true sentence or a false one (not both, not neither). Starting with a reflexive-free gappy logic of paradox, $\mathcal{CLP}^{\circ}$, we show that this logic can adequately capture paradoxicality and unparadoxicality. Moreover, we show that it is also immune to semantic paradoxes including revenge paradoxes. We then show that a non-transitive glutty logic of paradox, $\mathcal{CLP}^{\bullet}$, escapes problems of overinternalizations of semantic notions and subdues metainferential paradoxes. We end our discussion by showing that the two logics are equivalent—they are two sides of the same coin.

Keywords— Paradox; Revenge; Classical Recapture; Gappy; Glutty; Overinternalization

1 Introduction

In order to understand the semantic paradoxes, we will start with a simple example: The Liar, λ. The Liar is a sentence that says of itself that it is not true. This raises the question: Is the Liar true or false? Suppose that it is true. Then, what it says must be the case, but it is saying that it is not true. Hence, we reach a contradiction. Suppose that it is not true. Then, the Liar confirms it and must be telling the truth. Thus, it must be true, and so, once again, we reach a contradiction. The truth value of λ cannot consistently be true, nor can it consistently be false.

For this paper, we will let this example shape our understanding of what a (semantic) paradoxical sentence is. To make our point a bit more precise, let $\mathcal{P}\{1, 0\}$, the power set of $\{1, 0\}$, be a set of truth values, T, where 1 represents

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'truth' and 0 represents 'falsity'. The subsets $\{1\}$ and $\{0\}$ can, in turn, be understood as the particular values that, in principle, a formula in a language can be expected to receive.[1] Thus, given a language $\mathcal{L}$, a valuation v is a function $v : \mathcal{L} \to T$. Then:

Definition 1.1. A sentence φ is *paradoxical* if and only if, for every v, it is neither the case that $v(\varphi) = \{1\}$ nor $v(\varphi) = \{0\}$.

How are we to represent paradoxicality in our logic? We have two options. First, we can say that since φ is not just true and is not just false either, paradoxicality can be represented as neither true nor false, $v(\varphi) \notin \{1, 0\}$ (i.e., $v(\varphi) = \varnothing$). Alternatively, we can say that, since φ is not just true and is not just false, paradoxicality can be represented as being both true and false, $v(\varphi) = \{1, 0\}$. The first route admits truth-value gaps, following the footsteps of paracomplete approaches (e.g., [14]). The second admits truth-value gluts, following the footsteps of paraconsistent approaches (e.g., [27]).

In the present paper, we will present a particular way to show how systems that express these alternative views can ultimately align. More precisely, we will argue that, under certain specific conditions that will be discussed later, choosing between these two options may not be significantly different. Our results shed a new, interesting light on the relations between paraconsistency and paracompleteness. Furthermore, along the way of establishing such results, some interesting issues concerning semantic theories will arise.[2]

The paper is structured as follows. In the next section, we will represent paradoxicality as the 'neither' option—$v(\varphi) \notin \{1, 0\}$ and our logic, $\mathcal{CLP}^{\yang}$, will be a gappy reflexive-free logic. We will then present a theory of truth built on it in order to show how it avoids semantic paradoxes (including revenge paradoxes). It not only subdues these paradoxes, but can also express the facts that they are paradoxical. We will then present a glutty transitive-free logic, $\mathcal{CLP}^{\yin}$. The logic $\mathcal{CLP}^{\yin}$ represents paradoxicality as the 'both' option—$v(\varphi) = \{1, 0\}$. We will show how this logic avoids problems of overinternalization of semantic notions and subdues metainferential paradoxes. We will conclude by arguing that the two logics we present are equivalent in the sense that their provable/valid sequents can be interpreted to be saying the same thing. Additionally, every significant result of $\mathcal{CLP}^{\yang}$ is also present in $\mathcal{CLP}^{\yin}$ and vice versa. Thus, the logic $\mathcal{CLP}$, whether the gappy or glutty version, can express semantic notions such as truth, validity, and paradoxicality, can establish the paradoxicality of paradoxical sentences, subdues semantic paradoxes including metainferential paradoxes

[1] For simplicity, we will be omitting brackets when it comes to the truth-values. For instance, when presenting the truth tables below, we will employ merely 1 and 0, instead of $\{1\}$ and $\{0\}$ for 'just true' and 'just false' respectively.

[2] To be clear, some logicians have previously argued that gappy and glutty theories can be understood as providing symmetrical solutions to paradoxes [29]. Our point here, however, is different. Our analysis aims to establish why this is so; these approaches ultimately express the same semantic commitments. As a result, they can thus be unified in one and the same logic. Moreover, our analysis sheds light on the relation between gappy and glutty theories and substructural approaches to paradoxes. Thus, it provides an improvement on our understanding of the relations and implications of gaps and gluts.

and revenge paradoxes, and avoids issues of underinternalizations and overinternalizations.

2 $\mathcal{CLP}^{℗}$

Logics are usually understood in terms of their valid inferences. Recently, however, some argued that higher-order inferences—inferences between inferences—should also be considered, on pain of otherwise giving an incomplete picture of what a logic is [8].[3] Be that as it may, here we will consider that if several inferential levels are taken into account, then a logic should be consistently homogeneous at every level.[4] We here provide a homogeneous logic dealing with gaps.

Given what we said, when we define validity, we will define it in a uniform way for sequents and meta$_n$sequents. Moreover, we take that sequents and meta$_n$sequents can receive truth values just in the same way in which formulas can. Applying truth values to sequents is not unheard of. For example, in [28], Priest states that "There is, note, no problem about applying the truth predicate to sequents. These are, after all, simply meta-theoretic conditionals, with their own verification conditions" (fn. 41).[5]

Definition 2.1. A *sequent* is an ordered pair $\langle \Gamma, \Delta \rangle$ (which we will present as $\Gamma \Rightarrow \Delta$) where Γ and Δ are sets of formulas.

Definition 2.2. A *metasequent of level n* is an ordered pair $\langle \Gamma, \Delta \rangle$ (which we will write as $\Gamma \Rightarrow_n \Delta$) where Γ and Δ are sets of sequents of level $n-1$.

2.1 Model theory of $\mathcal{CLP}^{℗}$

We will use a standard language for propositional logic, augmented by a primitive operator $\mathbb{A}$. Informally, we can understand $\mathbb{A}\varphi$ as expressing that excluded middle does not hold for the corresponding formula, and thus, can be understood as $\neg(\varphi \vee \neg\varphi)$. In other words, $\mathbb{A}$ will represent paradoxicality in this logic.[6]

We will be relying on what we will call $\mathcal{CLP}^{℗}$-models where $\mathbb{A}$ is taken to be bivalent. In the following, the truth value n is a shorthand for $\notin \{1, 0\}$ (i.e., $= \{\}$):[7]

[3] This line of reasoning, however, has been contested. See, for example, Ripley's [32].

[4] Once again, this hasn't gone unchallenged. See, for instance, Barrio et al's [4]. We acknowledge such cases, but we will set them aside for the present purposes. Consistency is an assumption in our work.

[5] Even though we emphasize the valuational aspect with respect to inferences, conversely, formulas can be seen as having entailment-like properties. Thus, tautologies can be seen as validities, contingencies as invalidities, and contradictions as antivalidities. For more details, see Barrio and Pailos' [6].

[6] To anticipate, we will add a paradoxicality predicate that says of a sentence that it is paradoxical. However, such a predicate will not show up until we build a theory of truth in §2.3.

[7] Note that these tables are strong Kleene models [21] with an addition of a bivalent operator.

$\neg$	
1	0
n	n
0	1

$\varphi \vee \psi$	1	n	0
1	1	1	1
n	1	n	n
0	1	n	0

$\rightarrow$	1	n	0
1	1	n	0
n	1	n	n
0	1	1	1

$\wedge$	1	n	0
1	1	n	0
n	n	n	0
0	0	0	0

$\triangle$	
1	0
n	1
0	0

The reason that the $\triangle$ operator behaves bivalently in $\mathcal{CLP}^{\circlearrowleft}$ is because $\triangle$ expresses a meta-semantic commitment: paradoxical formulas are those for which excluded middle does not hold. Therefore, $\triangle$ provides us with a resource through which we can refer to the status of a given formula. If φ is paradoxical, $\triangle\varphi$ will hold, and thus $\triangle\varphi$ will be the case (i.e., it will be true). If φ is not paradoxical, $\triangle\varphi$ will not be the case (i.e., it will be false). This, in turn, will ensure that paradoxicality is a bivalent notion. If, however, one does not want to introduce a primitive operator to express paradoxicality, but rather define $\triangle\varphi$ as a literal shorthand for $\neg(\varphi \vee \neg\varphi)$, then there are two possible routes of doing so, which can be found in §5.

Notice also that, given the way we define $\triangle$, we can get constants for free. After all, $\triangle\varphi$ cannot receive the value n. Thus, a formula that says that a paradoxical sentence is paradoxical, $\triangle\triangle\varphi$, will always receive the value 0. Moreover, the negation of the former, $\neg\triangle\triangle\varphi$, will always receive the value 1. Thus, we can abbreviate $\triangle\triangle\varphi$ as $\bot$ and $\neg\triangle\triangle\varphi$ as $\top$ for any φ.

We are now in a position to define validity for sequents and metasequents:

Definition 2.3. A sequent, $\Gamma \Rightarrow \Delta$, is $\mathcal{CLP}^{\circlearrowleft}$-valid if and only if for all $\mathcal{CLP}^{\circlearrowleft}$-models, $\exists\gamma \in \Gamma$ s.t. $v(\gamma) = 0$ or (inclusive) $\exists\delta \in \Delta$ s.t. $v(\delta) = 1$.

Definition 2.4. A meta$_n$sequent, $\Gamma \Rightarrow_n \Delta$, is $\mathcal{CLP}^{\circlearrowleft}$-valid if and only if for all $\mathcal{CLP}^{\circlearrowleft}$-models, $\exists\gamma \in \Gamma$ s.t. $v(\gamma) = 0$ or (inclusive) $\exists\delta \in \Delta$ s.t. $v(\delta) = 1$.[8]

Thus, the logic is very close to the strong Kleene non-reflexive logic TS, presented by French in [17]. In fact, in both logics, Reflexivity ($\varphi \Rightarrow \varphi$) will fail. To see this, just take a valuation that assigns n to the formula. It is interesting to notice that on this way of presenting $\mathcal{CLP}^{\circlearrowleft}$ there is no need to understand consequence in terms of a "mixed relation" to account for such

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[8]Notice that meta$_n$sequents are being treated as conditionals of sorts. In the present framework, we treat valuations as assignments of truth values to the elements of a domain, including sequents of any level. In $v : \mathcal{L} \to T$, the domain includes sequents as 'higher-order' formulas.

failures of Reflexivity [10]. That is, we do not need to state consequence as involving two sets of designated values. Instead, we can simply take a set of designated values, $\{1\}$, and a set of undesignated values, $\{0\}$, and let validity be such that, if $v(\bigwedge \Gamma) \notin \{0\}$, then $v(\bigvee \Delta) \in \{1\}$.

Notice that in our framework there is still another way of understanding validity in $\mathcal{CLP}^{\yin}$. In fact, we can define validity by saying there is just one designated set, $D = \{1\}$, and that a sequent is valid if and only if $v(\Gamma \Rightarrow \Delta) \in D$, where $\Rightarrow$ is assessed in the same way that the conditional $\rightarrow$ is, the commas on the left read as conjunctions, and those on the right, as disjunctions. Thus, for $v(\Gamma \Rightarrow \Delta) = 1$, there must be a $\gamma \in \Gamma$ that receives value 0 or a $\delta \in \Delta$ that receives value 1, and this is just the definition of validity that we used.[9]

Thus, the failure of Reflexivity brings $\mathcal{CLP}^{\yin}$ and TS together, but there are differences between both approaches that are worth stressing. After all, these logics are defined through slightly different kinds of models, and such a difference has drastic consequences. Given $\mathcal{CLP}^{\yin}$ models, the logic can be seen as a classical recaptured version of the non-reflexive logic TS. Indeed, in our logic we are able to add $\mathbb{A}\varphi$ on the right. If a formula φ receives n, $\mathbb{A}\varphi$ will get the value 1, and so we can have a version of Reflexivity back as we will see shortly. This is a recapturing logic; meaning we can get classical reasoning back if we are explicit about the assumptions over our formulas.[10]

Thus, while TS is empty in the absence of truth-value constants, $\mathcal{CLP}^{\yin}$ is far from empty. Also, the considerations provided here carry over to every level in $\mathcal{CLP}^{\yin}$, given the way consequence is understood. Meanwhile, the TS logician might stay silent with regards to validity of higher levels.

The idea of establishing these comparisons is merely to shed some light on the relations between these non-Reflexive logics. The features distinctive of $\mathcal{CLP}^{\yin}$ will show their qualities shortly.

2.2 Proof theory of $\mathcal{CLP}^{\yin}$

Unsurprisingly, $\mathcal{CLP}^{\yin}$ lacks Reflexivity all the way up and not just on the first level, since such an inference is unfaithful to the assumed semantic landscape; given an understanding of sequents in which each side corresponds to a semantic

[9] In this sense, the consequence relation is a pure consequence rather than a mixed consequence. For more on pure and mixed consequence, see [10]. We will revisit the notion of pure consequence once we present $\mathcal{CLP}^{\yang}$.

[10] One might say that, since Reflexivity does not hold unrestrictedly, the concept of validity in play in this logic has a non-classical flavor. We call it "Classical Logic of Paradox", however, since it is a classical recapture logic, that is, a logic capable of recovering classical logic, which is also capable of handling and expressing paradoxicality. Our approach should, in any case, be distinguished from approaches that deviate radically from classicality when it comes to validity. In particular, recently, Weber [37] and Barrio et al. [5] have put forward approaches where non-classicality spreads further such that validity itself is taken to be a non-bivalent notion. For us, inferences are valid or invalid, and never both. Interestingly, however, these approaches, like ours, provide ways of obtaining a uniformity between logic and metalogic. As we will see, this has some beneficial consequences for us, such as allowing us to avoid revenge paradoxes. Of course, these considerations include the case of $\mathcal{CLP}^{\yang}$ as well. We thank an anonymous referee for inviting us to clarify this point.

truth-value, Reflexivity can be read as saying that, for any φ, φ is either true or false. However, this is inaccurate in our logic; φ can either be true or false, or even neither (in which case it is paradoxical).[11] The way in which we represent this fact proof-theoretically will be via $\triangle$ as shown below. To reiterate, $\mathcal{CLP}^{\circlearrowleft}$ acts as a recapturing version of TS. In other words, certain inferential patterns that were not available in TS can be recovered in $\mathcal{CLP}^{\circlearrowleft}$.

Moreover, $\triangle\triangle \Rightarrow$ is provable in our logic which, in line with what we said in the previous section, represents $\bot \Rightarrow$. With the help of our negation rule, $\Rightarrow \top$ is also provable. We now introduce a sequent-style presentation of this logic for the first inferential level:

I. Axioms

$$\frac{}{\triangle\varphi \Rightarrow \triangle\varphi}\ \triangle\text{-Reflexivity} \qquad \frac{}{\varphi \Rightarrow \varphi, \triangle\varphi}\ \Rightarrow\triangle$$

II. Operational rules

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma, \triangle\varphi \Rightarrow \Delta}\ \triangle\Rightarrow \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \triangle\varphi \Rightarrow \Delta}\ \triangle\Rightarrow$$

$$\frac{\Gamma, \triangle\varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \varphi, \Delta}\ \triangle\Rightarrow^{\uparrow}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\ \neg\Rightarrow \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\ \Rightarrow\neg$$

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ \wedge\Rightarrow \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ \Rightarrow\wedge$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\ \Rightarrow\vee \qquad \frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\ \vee\Rightarrow$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ \Rightarrow\to \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\ \to\Rightarrow$$

III. Structural rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ \text{Weakening} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \text{Weakening}$$

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ \text{Contraction}\Rightarrow \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ \Rightarrow\text{Contraction}$$

[11] In his pioneering paper, French [17] puts the point in terms of acceptance and rejection. Identity, he says, tells us that if a sentence is not rejected, then it should be accepted. Thus, "reflexivity will fail for any sentence C which one should neither reject nor accept"(p.122).

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{Cut}$$

The double horizontal lines indicate that the rule is invertible (i.e., it can be flipped). For rules that require two premises going from up to down, such as $\vee \Rightarrow$, reading the invertible rule simply splits into two rules, one for each premise acting as the conclusion. As for $\triangle \Rightarrow^{\uparrow}$, it is the invertible rule of $\triangle \Rightarrow$.[12]

As for higher levels, they follow suit in the sense that, given what we said above, we should simply swap $\Rightarrow$ with $\Rightarrow_i$, and let φ and ψ stand for sequents of level $i-1$, and Γ and Δ stand for sets of sequents of level $i-1$.[13]

For example, if we want to state the right Weakening rule for all levels, we would write:

$$\frac{\Gamma_{i-1} \Rightarrow_i \Delta_{i-1}}{\Gamma_{i-1} \Rightarrow_i \Delta_{i-1}, \varphi_{i-1}}$$

That means if $i = 0$—the inferential level—then Γ_{-1} and Δ_{-1} are sets of formulas and φ_{-1} is a formula. Whereas, if $i \geq 1$, then Γ_{i-1} and Δ_{i-1} are sets of sequents of level $i-1$ and φ_{i-1} is a sequent of level $i-1$. However, for legibility, we opted for presenting the rules of the first level. The reader can use these remarks to have a grip on what higher levels look like.[14]

2.3 Theory of Truth

We expand our language to include a truth predicate, $Tr(\langle\varphi\rangle)$ where $\langle\rangle$ is a naming device as in [30]. Semantically, $v(Tr(\langle\varphi\rangle)) = v(\varphi)$. Proof theoretically, we add the following rules:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, Tr(\langle\varphi\rangle) \Rightarrow \Delta}\ \text{L-Transparency} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow Tr(\langle\varphi\rangle), \Delta}\ \text{R-Transparency}$$

[12] Also, note that $\triangle$-Reflexivity is derivable from $\Rightarrow \triangle$, $\triangle \Rightarrow$, and Contraction.

[13] Meta sequents are commonly presented through single conclusions. However, the framework adopted here–where these are understood as conditionals of sorts–makes it natural to allow for multiple conclusions. After all, conditionals can have disjunctions as their consequents. Notice, more generally, that there is, in principle, nothing wrong with multiple conclusions concerning metasequents: these can be understood as merely saying that at least one of the consequent sequents follows from a given set of sequents. Such an understanding, moreover, is not without precedents. For instance, Ahmad [1], Porter [25] (e.g., p. 86), and Priest [28] explicitly use multiple conclusions when presenting higher-order paradoxes and allow logical operations apply to sequents and meta$_n$sequents. Moreover, talking of metainferences as inferences is not uncommon – see, for instance, [24]. But multiple conclusions are commonly accepted in the case of inferences, making the generalization of multiple conclusions in the case of metasequents compelling. Notice, moreover, that the way of obtaining higher levels makes logical operations (e.g., negation) apply to sequents of any level as well. The idea here endorsed makes this natural. Although admittedly non-standard in general, there appears to be nothing preventing one from doing so. Many thanks to the anonymous reviewer for urging us to elaborate on our position.

[14] Even though throughout the paper we use sets instead of multisets, this choice is merely to simplify the completeness proof below. We could have used multisets instead since Contraction is admissible in $\mathcal{CLP}^{\wp}$. The avid reader can refer to the admissibility of Contraction in ([20] pp. 117-121) for the logic $G3$. All of the relevant parts for the proof of the admissibility of Contraction for $G3$ are shared with $\mathcal{CLP}^{\wp}$.

Moreover, we will enable, as part of our theory, the move from the operator $\triangle\varphi$ to the corresponding claim $Par(\langle\varphi\rangle)$ (i.e., 'φ is paradoxical'). Again, informally, one can understand this as expressing something along the lines of $\neg(Tr(\langle\varphi\rangle) \vee \neg Tr(\langle\varphi\rangle))$. So, semantically, $v(Par(\langle\varphi\rangle) = 1$ iff $v(\varphi) = n$, $v(Par(\langle\varphi\rangle) = 0$ otherwise. Proof theoretically, we add the following two rules:

$$\frac{\Gamma, \triangle\varphi \Rightarrow \Delta}{\Gamma, Par(\langle\varphi\rangle) \Rightarrow \Delta}\ \text{L-Paradoxicality} \qquad \frac{\Gamma \Rightarrow \triangle\varphi, \Delta}{\Gamma \Rightarrow Par(\langle\varphi\rangle), \Delta}\ \text{R-Paradoxicality}$$

To illustrate how our logic handles paradoxes, we will begin with the Liar. Let λ be equivalent to $\neg Tr(\langle\lambda\rangle)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\lambda \Rightarrow \lambda, \triangle\lambda}\ {\Rightarrow}\triangle}{Tr(\langle\lambda\rangle) \Rightarrow \lambda, \triangle\lambda}\ \text{L-Transparency}}{\Rightarrow \neg Tr(\langle\lambda\rangle), \lambda, \triangle\lambda}\ {\Rightarrow}\neg}{\Rightarrow \lambda, \lambda, \triangle\lambda}\ \text{Def of } \lambda}{\Rightarrow \lambda, \triangle\lambda}\ \text{Contraction} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\lambda \Rightarrow \lambda, \triangle\lambda}\ {\Rightarrow}\triangle}{\lambda \Rightarrow Tr(\langle\lambda\rangle), \triangle\lambda}\ \text{R-Transparency}}{\neg Tr(\langle\lambda\rangle), \lambda \Rightarrow \triangle\lambda}\ \neg{\Rightarrow}}{\lambda, \lambda \Rightarrow \triangle\lambda}\ \text{Def of } \lambda}{\lambda \Rightarrow \triangle\lambda}\ \text{Contraction}}{\dfrac{\Rightarrow \triangle\lambda}{\Rightarrow Par(\langle\lambda\rangle)}\ \text{R-Paradoxicality}}\ \text{Cut}$$

Thus, not only does the paradox not trivialize our theory, but our theory is able to capture the paradoxicality of the Liar.

It is important to note that the move Def of λ, here, is a shortcut. For instance, the full move of Def of λ on the right side should be the following:

$$\dfrac{\dfrac{\dfrac{}{\lambda \Rightarrow \neg Tr(\langle\lambda\rangle), \triangle\lambda}\ \text{Id Est} \qquad \dfrac{\vdots}{\neg Tr(\langle\lambda\rangle), \lambda \Rightarrow \triangle\lambda}}{\lambda, \lambda \Rightarrow \triangle\lambda, \triangle\lambda}\ \text{Cut}}{\dfrac{\lambda, \lambda \Rightarrow \triangle\lambda}{\vdots}}\ \text{Contraction}$$

So any application of "Def of φ" move would use a form of Reflexivity (e.g., $\Rightarrow \triangle$ in $\mathcal{CLP}^{?}$) and a form of Cut.[15] So, whenever such a move appears in a derivation, it inserts a $\triangle$ on the right.

Furthermore, to avoid exhausting the reader with several derivations that show how the theory handles the usual semantic paradoxes, it suffices to note that a Curry sentence with a false consequent can be proven to be paradoxical, a Curry with a true consequent can be proven to be true and unparadoxical,

[15]To anticipate, in $\mathcal{CLP}^{\bullet}$, we have Reflexivity and a form of Cut, $\Rightarrow \triangle$, in a "Def of" move where $\triangle$ is introduced in the consequent of the conclusion sequent. This follows Hlobil's observation where he notes in [19] (p. 4, fn. 6), "It is not obvious, however, that in a nontransitive setting, the diagonal lemma suffices to guarantee the intersubstitutability of π and $Val(\langle\pi\rangle, \langle A\rangle)$".

and a Curry with a contingent consequent stays neutral as it should. After all, neutrality is the name of the game.

We will turn to soundness and completeness for the theory of truth based on $\mathcal{CLP}^{\wp}$, which we will call $t\mathcal{CLP}^{\wp}$. Given our previous considerations concerning the hierarchical nature of the logic and its uniformity, the following result covers sequents of any finite level. To make this explicit, we label the arrow with a subscript alluding to the level $n \in \omega$ at which the sequent belongs.

Theorem 2.1 (Soundness and Completeness). $\models_{t\mathcal{CLP}^{\wp}} \Gamma_n \Rightarrow_n \Delta_n$ if and only if $\vdash_{t\mathcal{CLP}^{\wp}} \Gamma_n \Rightarrow_n \Delta_n$.

Proof. The *soundness* proof is straightforward and is left to the avid reader. As for *completeness*, the proof can be found in the appendix. □

2.4 $\mathcal{CLP}^{\wp}$ is Revenge Free

We have previously shown how our theory can internalize the notion of paradoxicality. Given our use of the corresponding predicate, one concern is that revenge paradoxes may arise. In fact, the menace of revenge paradoxes arising from the use of a paradoxicality predicate has recently prompted much attention.

Revenge paradoxes are paradoxes that use concepts involved in the diagnosis that some theories provide concerning the usual paradoxes. Murzi and Rossi [23] argued that paradoxicality is a central concept for non-classical theories, which turns out to be inexpressible for some on pain of revenge.[16] We will begin with an example that illustrates how revenge arguments are also subdued. Let ζ be equivalent to $\neg Tr(\langle\zeta\rangle) \vee Par(\langle\zeta\rangle)$:

$$
\begin{array}{ll}
\overline{\zeta \Rightarrow \zeta, \triangle\zeta} & \Rightarrow \triangle \\
\overline{Tr(\langle\zeta\rangle) \Rightarrow \zeta, \triangle\zeta} & \text{L-Transparency} \\
\overline{\Rightarrow \zeta, \neg Tr(\langle\zeta\rangle), \triangle\zeta} & \Rightarrow \neg \\
\overline{\Rightarrow \zeta, \neg Tr(\langle\zeta\rangle), Par(\langle\zeta\rangle)} & \text{R-Paradoxicality} \\
\overline{\Rightarrow \zeta, \neg Tr(\langle\zeta\rangle) \vee Par(\langle\zeta\rangle)} & \Rightarrow \vee \\
\overline{\Rightarrow \zeta, \zeta, \triangle\zeta} & \text{Def of } \zeta \\
\Rightarrow \zeta, \triangle\zeta & \text{Contraction}
\end{array}
$$

Call this derivation D_0

[16] In fact, what Murzi and Rossi showed is that certain paracomplete, paraconsistent, non-contractive, and non-transitive theories suffer from expressive limitations when it comes to paradoxicality. Rosenblatt [34] raises some concerns regarding the arguments in Murzi and Rossi's work, mainly related to the alleged relevance of the concept of paradoxicality for non-classical logicians and the justification for the rules they appeal to in their work. It is important to observe, moreover, that as made clear in [2], Murzi and Rossi's arguments depend upon the logics following a Strong Kleene schema, since Weak Kleene logics can be free of such paradoxes. We here show that to avoid revenge paradoxes arising from paradoxicality, Weak Kleene is not mandatory.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_0}{\Rightarrow \zeta, \triangle\zeta}}{\Rightarrow Tr(\langle\zeta\rangle), \triangle\zeta}\,\text{R-Transparency}}{\neg Tr(\langle\zeta\rangle) \Rightarrow \triangle\zeta}\,\neg\Rightarrow \qquad \dfrac{\dfrac{\dfrac{D_0}{\Rightarrow \zeta, \triangle\zeta}}{\triangle\zeta \Rightarrow \triangle\zeta}\,\triangle\Rightarrow}{Par(\langle\zeta\rangle) \Rightarrow \triangle\zeta}\,\text{L-Paradoxicality}}{\neg Tr(\langle\zeta\rangle) \vee Par(\langle\zeta\rangle) \Rightarrow \triangle\zeta}\,\vee\Rightarrow}{\zeta \Rightarrow \triangle\zeta, \triangle\zeta}\,\text{Def of } \zeta}{\zeta \Rightarrow \triangle\zeta}\,\text{Contraction}
$$

Call this derivation D_1

$$
\dfrac{\dfrac{D_0}{\Rightarrow \zeta, \triangle\zeta} \qquad \dfrac{D_1}{\zeta \Rightarrow \triangle\zeta}}{\Rightarrow \triangle\zeta}\,\text{Cut}
$$

Thus, not only does our theory subdue the revenge paradox, but it also establishes its paradoxicality.[17]

One might worry that if we let a sentence that says of itself that it is paradoxical, or if it says of itself that it is unparadoxical, then that might create some kind of revenge. Fortunately, these sentences are not paradoxical; the former is false and unparadoxical, the latter is true and unparadoxical, and our system internalizes these facts. To see this, let θ be equivalent to $Par(\langle\theta\rangle)$ and let μ be equivalent to $\neg Par(\langle\mu\rangle)$. To simplify the proof, we will assume full intersubstitutability between $\triangle$ and $Par(\langle\rangle)$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mu \Rightarrow \neg Par(\langle\mu\rangle), \triangle\neg Par(\langle\mu\rangle)}\,\text{id est } \mu \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\triangle\mu \Rightarrow \triangle\mu}\,\triangle\text{-Reflexivity}}{\triangle\mu, \neg\triangle\mu \Rightarrow}\,\neg\Rightarrow}{\neg\triangle\mu \Rightarrow \neg\triangle\mu}\,\Rightarrow\neg}{\triangle\neg\triangle\mu \Rightarrow \neg\triangle\mu}\,\triangle\Rightarrow}{\triangle\neg\triangle\mu, \triangle\neg\triangle\mu \Rightarrow}\,\triangle\Rightarrow}{\triangle\neg\triangle\mu \Rightarrow}\,\text{Contraction}}{\triangle\neg Par(\langle\mu\rangle) \Rightarrow}\,\text{L-Paradoxicality}}{\mu \Rightarrow \neg Par(\langle\mu\rangle)}\,\text{Cut}}{\triangle\mu \Rightarrow \neg Par(\langle\mu\rangle)}\,\triangle\Rightarrow}{Par(\langle\mu\rangle) \Rightarrow \neg Par(\langle\mu\rangle)}\,\text{L-Paradoxicality}}{\Rightarrow \neg Par(\langle\mu\rangle), \neg Par(\langle\mu\rangle)}\,\Rightarrow\neg}{\Rightarrow \neg Par(\langle\mu\rangle)}\,\text{Contraction}
$$

[17] Rosenblatt [34] introduces a fixed-point semantics for a language that can express the non-classical logician's notion of paradoxicality. Rosenblatt's theory remains silent on the semantic status of a revenge sentence such as ζ, meaning that it neither asserts that it is paradoxical nor that it is not. In contrast, our theory establishes that the revenge sentence we are dealing with is paradoxical.

The proof of $Par(\langle\theta\rangle) \Rightarrow$ is not very different. Hence, μ is logically equivalent to $\top$, and θ is logically equivalent to $\bot$. We would like to call μ a 'truth-teller' but, alas, that label is reserved for hypodoxes.

In a sense, our logical system categorizes its sentences into four different categories:

- Paradoxical sentences: where if φ is paradoxical, $\triangle\varphi$ turns out to be true — informally "$(\varphi \vee \neg\varphi)$" fails, (e.g., λ).
- Logically false sentences: sentences of the form $\triangle\triangle\varphi$ (i.e., $\bot$)(e.g., θ).
- Logically true sentences: sentences that are of the form $\neg\triangle\triangle\varphi$ (i.e., $\top$)(e.g., μ).
- Contingent sentences: sentences such as φ (e.g., hypodoxes like the truth-teller where τ is equivalent to $Tr(\langle\tau\rangle)$)).

Showing that some instances of the usual revenge paradoxes are subdued is not enough. In fact, we can show that there can be no revenge paradoxes in our system. Since every sequent in $\mathcal{CLP}^{\circlearrowleft}$ will contain at least one $\triangle$, and every sequent in $t\mathcal{CLP}^{\circlearrowleft}$ will contain either a $\triangle$ or, equivalently, a formula with a paradoxicality predicate, triviality is avoided.

To make this more precise, say that a formula φ is a subformula of ψ if it appears in its construction tree, as standardly understood. Note that every formula is a subformula of itself. We denote the set of subformulas of ψ with $Sub(\psi)$. Then, we have the following:

Lemma 2.2. For every $CLP^{\circlearrowleft}$-derivable sequent, $\vdash_{CLP^{\circlearrowleft}} \Gamma \Rightarrow \Delta$ and some formula $\psi \in \Gamma \cup \Delta$, it holds that $\triangle\varphi \in Sub(\psi)$ for some φ. Consequently, for every sequent $\vdash_{tCLP^{\circlearrowleft}} \Gamma \Rightarrow \Delta$ (i.e., which is derivable in the theory of truth based on $\mathcal{CLP}^{\circlearrowleft}$) and $\psi \in \Gamma \cup \Delta$, it holds that $\triangle\varphi \in Sub(\psi)$ or, equivalently, that $Par(\varphi) \in Sub(\psi)$.

Proof. Every axiom starts with $\triangle$. In order to get rid of $\triangle$, there must be a paradoxical sentence that establishes its non-paradoxicality. That requires sentences that use the paradoxicality predicate. As a result, $\triangle$ is merely replaced by a paradoxicality predicate, but that predicate is not eliminable without reintroducing $\triangle$ since it requires an id est axiom and Cut. □

Theorem 2.3. The theory of truth based on $\mathcal{CLP}^{\circlearrowleft}$ is revenge-free.

Proof. This follows straightforwardly from Lemma 2.2. □

Having a non-trivial theory capable of dealing with several logical puzzles is attractive. However, one might feel uneasy concerning the apparent costs of the present approach. After all, non-triviality is ensured by structuring the axioms in such a way that provable sequents will make some reference to paradoxicality.[18]

[18] We thank an anonymous referee for raising this concern.

We take logic to be neutral, and so logic should not be able to prove or disprove sentences except what is logically true, logically false, or establish the paradoxicality of logically paradoxical sentences. Consider the case of classical logic; it cannot prove or disprove formulas unless they are logically true (tautologies) or logically false (contradictions). The only difference between $\mathcal{CLP}$ and classical logic is that $\mathcal{CLP}$ acknowledges the possibility that some sentences are paradoxical, while classical logic, in a sense, is unaware of them.

In classical logic, one can establish contingent truths if and only if there are imported assumptions. In a similar fashion, in $\mathcal{CLP}$, the reference to paradoxicality can be removed from non-paradoxical inferences by importing assumptions conveying precisely that the domain, or the formulas involved, are safe. This is the difference between describing what logic is (as topic-neutral) and what can be applied to, concerning different domains. So, we believe that there is a case for the idea that there is no such cost.

One might, however, still press that if we check some revenge sentences, as ζ, model theoretically, then surely there is revenge for our logic. For instance:

1. Suppose $v(\zeta) = 1$, then $v(Par(\zeta)) = 0$ and $v(\neg Tr(\zeta)) = 0$, and so $v(\zeta) = 0$, thus we reach a contradiction.
2. Suppose $v(\zeta) = 0$, then $v(Par(\zeta)) = 0$ but $v(\neg Tr(\zeta)) = 1$, and so $v(\zeta) = 1$, thus we reach a contradiction.
3. Finally, suppose $v(\zeta) = n$, then $v(Par(\zeta)) = 1$, and so $v(\zeta) = 1$, thus we reach a contradiction.

This is a fallacious reasoning. The logic tells you that if a sentence cannot consistently receive the value 1 and cannot consistently receive the value 0, then it is paradoxical. So, from 1 and 2, we can establish that it is paradoxical. 3, then, is the fallacious part of the reasoning. The logic tells you that identity holds iff the sentence is not paradoxical (can get a consistent value of 1 or 0), otherwise identity does not hold and no further reasoning can be applied to it.

In other words, instances of Reflexivity, which can be thought of as expressing identity, hold in our logic just in case the formulas in it are not paradoxical. Moreover, recall that Reflexivity is the point of departure for our derivations. If ζ is paradoxical, as established by 1 and 2 above, then we cannot simply rely on the alleged identity between ζ and $\neg Tr(\langle\zeta\rangle) \vee Par(\langle\zeta\rangle)$ to establish a contradiction. After all, such a formula lacks the status that would enable us to stick to the equivalence. The real problem, we contend, is treating a paradoxical formula as well-behaved when it is not.[19]

So, the fallacy is basically applying Def of ζ metalogically, without realizing that there is an application of full Reflexivity which the logic rejects. This is to say, the apparent model-theoretic revenge paradox is caused by taking metalogic to be that of classical logic, one in which Reflexivity holds all across the board, and so, despite what is being established in 1 and 2, the identity in 3 is to be

[19]From another perspective, examples like ζ can be seen as evidence of why reflexivity can fail with regard to paradoxical sentences.

taken at face value. Once the metalogic is also $\mathcal{CLP}^{\circledcirc}$ (and logic and metalogic should be one and the same), the revenge paradox is subdued.

This might be a bit clearer in $\mathcal{CLP}^{☯}$. As we will see in more detail shortly, $\mathcal{CLP}^{☯}$ does not have Cut, but a Cut-like rule that introduces paradoxicality. Hence, when we say if $v(\zeta) = n$ then $v(Par(\zeta)) = 1$, we cannot conclude that $v(\zeta) = 1$ because that requires a Cut using an id est axiom. Since we want our metalogic to match our logic, we can only perform the Cut-like move and conclude $v(\zeta) = n$. Hence, there is no contradiction if our metalogic is the same as our logic.

3 $\mathcal{CLP}^{☯}$

3.1 Model theory of $\mathcal{CLP}^{☯}$

We will use a standard language for propositional logic. This time, we will use $\triangle\varphi$ to abbreviate $\varphi \wedge \neg\varphi$. As before, $\triangle$ will represent paradoxicality in this logic. We will be relying on $\mathcal{CLP}^{☯}$-models as shown in the tables below, where the truth value b is a shorthand for $= \{1, 0\}$:

$\neg$	
1	0
b	b
0	1

$\wedge$	1	b	0
1	1	b	0
b	b	b	0
0	0	0	0

$\vee$	1	b	0
1	1	1	1
b	1	b	b
0	1	b	0

$\rightarrow$	1	b	0
1	1	b	0
b	1	b	b
0	1	1	1

$\varphi \vee \neg\varphi$	
1	1
b	1
0	1

$\varphi \rightarrow \varphi$	
1	1
b	1
0	1

$\triangle$	
1	0
b	1
0	0

Definition 3.1. A sequent, $\Gamma \Rightarrow \Delta$, is $\mathcal{CLP}^{☯}$-valid if and only if for all $\mathcal{CLP}^{☯}$-models, $v(\bigwedge \Gamma \rightarrow \bigvee \Delta) = 1$.

Definition 3.2. A meta$_n$sequent, $\Gamma \Rightarrow_n \Delta$, is $\mathcal{CLP}^{☯}$-valid if and only if for all $\mathcal{CLP}^{☯}$-models, $v(\bigwedge \Gamma \rightarrow \bigvee \Delta) = 1$.

We can immediately observe that Cut fails in $\mathcal{CLP}^{☯}$, and thus, it is a non-transitive logic:

$$\frac{\overbrace{\overbrace{\Gamma}^{1} \Rightarrow \overbrace{\varphi}^{b}}^{b} \qquad \overbrace{\overbrace{\varphi}^{b} \Rightarrow \overbrace{\Delta}^{0}}^{b}}{\underbrace{\Gamma \Rightarrow \Delta}_{0}}$$

By the definition of metasequent validity in $\mathcal{CLP}^{\bullet}$, given that the inference takes one from b premises to 0 conclusion, it is metainferentially invalid.[20]

$\mathcal{CLP}^{\circ}$ can be understood as employing a mixed consequence as well as a pure consequence. $\mathcal{CLP}^{\bullet}$ can be seen as employing a pure consequence as well because, in our framework, the validity of the sequent is equivalent to the sequent being true (see [10]). As it happens, such a definition amounts to the failure of unrestricted Cut in the logic. Such a failure brings $\mathcal{CLP}^{\bullet}$ close to the logic ST. However, as in our previous case, there are still several differences worth stressing.[21]

To begin with, $\mathcal{CLP}^{\bullet}$ and ST are defined through different sorts of consequence relations and models. Interestingly, whereas $\mathcal{CLP}^{\bullet}$ can be seen as employing a pure consequence, a similar approach is not available to the Cut-free logic ST. Let us explain.

If we take validity to be equivalent to the truth of a sequent, then the theory of truth based on ST would be trivial, since Cut would be valid. What makes Cut invalid is the case in which the Cut-formula receives value b, which enables one to go from valid premises to an invalid conclusion (as the counterexample to Cut shown above). But if the validity of a metainference is understood in terms of the truth-values of sequents, then we go from b premises to 0 conclusion, which is valid in ST. So, the ST theorist cannot have a pure consequence—the same way we do—because their metaconsequence relation is bound to be different from their consequence relation on pain of triviality.[22]

Given the way the logic is defined, $\mathcal{CLP}^{\bullet}$ can be seen as a recapturing version

[20] In fact, there are instances of Cut that can take you from valid sequents to an invalid sequent; given the assumption that λ is identical to $\neg Tr(\langle\lambda\rangle)$:

$$\dfrac{\overbrace{\overbrace{\Gamma}^{1} \Rightarrow \overbrace{\lambda, \neg\lambda}^{1}}^{1} \qquad \overbrace{\overbrace{\lambda}^{b} \Rightarrow \overbrace{\neg Tr(\lambda)}^{b}}^{1 \text{ by assumption}}}{\underbrace{\Gamma \Rightarrow \neg\lambda, \neg Tr(\lambda)}_{b}}$$

[21] For more details on the logic ST, see [11, 12, 30, 31].

[22] We suspect that any attempt to remedy this issue—to have a pure consequence reading or to treat $\rightarrow$ and $\Rightarrow$ on an equal footing—would bring the ST-theorist closer to our $\mathcal{CLP}^{\bullet}$.

of a non-transitive logic at every level.[23] In particular, given the $\triangle$, the logic can be seen as having restricted forms of Cut. Moreover, as we are about to see, the $\triangle$ enables one to address some issues affecting non-transitive logics such as ST in a novel way.

The considerations here are not meant to move the advocate of ST towards $\mathcal{CLP}^{\rightmoon}$. In the end, the reader is to take their own conclusion on these matters. Rather, our point here is just to highlight the relations between two non-transitive logics.

3.2 Proof theory of $\mathcal{CLP}^{\rightmoon}$

Even though $\mathcal{CLP}^{\rightmoon}$ loses unrestricted Cut, it has a limited form of it and full Reflexivity:[24]

I. Axioms

$$\frac{}{\varphi \Rightarrow \varphi}\ \text{Reflexivity}$$

II. Operational Rules

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \triangle\varphi} \Rightarrow\triangle \qquad \frac{\Rightarrow \varphi}{\triangle\varphi \Rightarrow} \triangle\Rightarrow \qquad \frac{\varphi \Rightarrow}{\triangle\varphi \Rightarrow} \triangle\Rightarrow$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta}{\neg\varphi, \Gamma \Rightarrow \Delta, \triangle\varphi} \neg\Rightarrow \qquad \frac{\Gamma \Rightarrow \triangle\varphi, \Delta}{\neg\triangle\varphi, \Gamma \Rightarrow \Delta} \neg\triangle\Rightarrow \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi} \Rightarrow\neg$$

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\Rightarrow \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \Rightarrow\wedge$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \Rightarrow\vee \qquad \frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \vee\Rightarrow$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi} \Rightarrow\to \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta, \triangle\varphi, \triangle\psi} \to\Rightarrow$$

[23] It is not quite ST though since it is non-transitive even at the inferential level:

$$\frac{\overbrace{\overset{b}{\overbrace{\varphi}} \Rightarrow \overset{b}{\overbrace{\varphi}}}^{1} \qquad \overbrace{\overset{0}{\overbrace{\psi}} \Rightarrow \overset{0}{\overbrace{\psi}}}^{1}}{\underbrace{\underbrace{\varphi}_{b}, \underbrace{\varphi \to \psi}_{b} \Rightarrow \underbrace{\psi}_{0}}_{b}}$$

Similar counterexamples for higher levels apply. Observe that the presence of $\triangle$s on the right-hand side of the sequent arrow would restore transitivity, in a similar fashion to the way in which Reflexivity is handled through $\triangle$s in $\mathcal{CLP}^{\wp}$.

[24] For reasons that will become apparent in §4, we will not provide proofs of soundness and completeness for $\mathcal{CLP}^{\rightmoon}$ here.

III. Structural rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta, \mathbb{A}\Gamma, \mathbb{A}\Delta}\text{ Weakening} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi, \mathbb{A}\Gamma, \mathbb{A}\Delta}\text{ Weakening}$$

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\text{ Contraction} \Rightarrow \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} \Rightarrow \text{Contraction}$$

$$\frac{\Gamma \Rightarrow \Delta, \mathbb{A}\varphi \qquad \mathbb{A}\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathbb{A}\text{Cut}$$

$\mathcal{CLP}^{\bullet}$'s proof theory may seem convoluted at first, especially due to the left negation, weakening, and left conditional rules. However, notice that the restrictions that they impose are there for our own good. Indeed, they enable us to ensure that we do not prove, for every sentence, that the sentence is unparadoxical; they allow us, moreover, to treat $\mathbb{A}$ bivalently, and, furthermore, they ensure that every sequent expresses, for at least one sentence, that it is either true, false, or both.

Observe, moreover, that $\mathbb{A} \Rightarrow$ moves suggest that we can introduce a conjunction of a sentence with its negation only if there are no contexts. In other words, we can prove that a sentence is not paradoxical if and only if we can prove that such a sentence is just true or if we can prove that such a sentence is just false.

The reason 'Cut', $\Rightarrow \mathbb{A}$, is now behaving the way it does is the following: In classical logic, the rule of Cut can be seen as encoding the assumption that a sentence cannot be both true and false (with possible context). In the present case, to acknowledge that such cases are possible, we conclude from Cut that, since we prove the truth and prove the falsity of a given sentence (with possible context), then it must be both true and false. Thus, that is the reason why Cut has a conjunctive flavor when it comes to a sentence and its negation.[25] Nevertheless, since there are some sentences where their bivalence is established in $\mathcal{CLP}^{\bullet}$, we can have a limited Cut as shown by the rule $\mathbb{A}$Cut.

It is clear from our model theory and proof theory that we diverge a little from the non-transitive logic ST whereas we are more confined than it. This, however, is not a disadvantage of $\mathcal{CLP}^{\bullet}$. Other than the reasons provided earlier, if we take $\varphi \wedge \neg\varphi$ as saying that φ is paradoxical in ST, then it follows that every paradoxical sentence is also not paradoxical, because we can prove every sentence is not paradoxical in ST by $\neg \Rightarrow$ and $\wedge \Rightarrow$ on Reflexivity. Therefore, the Liar paradox is both paradoxical and not paradoxical in ST. In $\mathcal{CLP}^{\bullet}$, however, we take paradoxicality to be a bivalent notion. Therefore, the paradoxical sentences are said to be just paradoxical. Only those sentences for which we can prove their truth or their falsity (without context) can be said to be unparadoxical.

[25] Because it can be seen as an abbreviation of a right negation and a right conjunction moves, we decided to treat this version of Cut as an operational rule rather than as a structural rule.

3.3 Metainferential Paradoxes and $\mathcal{CLP}^{\bullet}$

Priest [28] and Porter [25] launched an attack on the substructural theorists such as advocates of ST claiming that even if the substructuralist handles semantic paradoxes by rejecting a structural rule, the substructuralist would soon face a metainferential 'revenge' where no metainferential structural rules are employed, but rather higher level ones are.

We will show that this does not occur in $\mathcal{CLP}^{\bullet}$. In fact, $\mathcal{CLP}^{\bullet}$ can establish that these revenge sequents are paradoxical. We will only show how it works for the metainferential Liar, but the same applies to other paradoxes presented, such as the metavalidity Curry.

We expand our language to include a fully transparent truth predicate for sequents. We will use $\Rrightarrow$ for $\Rightarrow_1$, Γ and Δ are sets of sequents, and ε is a sequent:

$$\dfrac{\Gamma \Rrightarrow \Delta, \varepsilon}{\Gamma \Rrightarrow \Delta, \Rightarrow Tr(\langle\varepsilon\rangle)}\ \text{R-Transparency}_1 \qquad \dfrac{\Gamma, \varepsilon \Rrightarrow \Delta}{\Gamma, \Rightarrow Tr(\langle\varepsilon\rangle) \Rrightarrow \Delta}\ \text{L-Transparency}_1$$

Let η be equivalent to $\neg(\Rightarrow Tr(\langle\eta\rangle))$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\eta \Rrightarrow \eta}\ \text{Reflexivity}_1}{\Rightarrow Tr(\langle\eta\rangle) \Rrightarrow \eta}\ \text{L-Transparency}_1}{\Rrightarrow \neg(\Rightarrow Tr(\langle\eta\rangle)), \eta}\ \Rrightarrow\neg}{\Rrightarrow \eta, \eta, \triangle\eta}\ \text{Def of }\eta}{\Rrightarrow \eta, \triangle\eta}\ \text{Contraction}_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\eta \Rrightarrow \eta}\ \text{Reflexivity}_1}{\eta \Rrightarrow \Rightarrow Tr(\langle\eta\rangle)}\ \text{R-Transparency}_1}{\neg(\Rightarrow Tr(\langle\eta\rangle)), \eta \Rrightarrow \triangle\eta}\ \neg\Rrightarrow}{\eta, \eta \Rrightarrow \triangle\eta, \triangle\eta}\ \text{Def of }\eta}{\eta \Rrightarrow \triangle\eta}\ \text{Contraction}_1\text{x2}}{\dfrac{\Rrightarrow \triangle\eta, \triangle\eta}{\Rrightarrow \triangle\eta}\ \text{Contraction}_1}\ \Rrightarrow\triangle$$

Hence, we proved that the sequent η is paradoxical. This is unsurprising, since our logic is the same metainferentially as it is inferentially. So in a sense, this solution is not very different from the one presented by Ahmad in [1] in collapsing the hierarchy, yet without actually collapsing the hierarchy; we just ensured that the outcome does not depend on the level at which you work.

Similarly, in [17] (pp. 126-127), French presents a metainferential validity Curry that leads to triviality in his system. In our system, whether the $\mathcal{CLP}^{\bullet}$ version or the one closer to French's system $\mathcal{CLP}^{\mathscr{C}}$, we can prove the paradoxicality of the metainferential validity Curry in addition to subduing it.

3.4 Overinternalization

A theory that is immune to semantic paradoxes is something appealing. However, theories can be expected to provide us with more than that. In particular, Barrio et al. [7] and Rosenblatt [33] posed the question as to what extent theories can adequately express their underlying concept of validity.

Let a theory of validity be a theory whose language is augmented with a predicate $Val(\langle\varphi\rangle, \langle\psi\rangle)$. What makes this a predicate for validity, from a proof-theoretic perspective, is that it is governed by the following pair of rules:

$$\frac{\varphi \Rightarrow \psi}{Val(\langle\varphi\rangle, \langle\psi\rangle)}\ \text{VP} \qquad \frac{}{\varphi, Val(\langle\varphi\rangle, \langle\psi\rangle) \Rightarrow \psi}\ \text{VD}$$

Informally, the first rule says that if an inference holds, then we can prove that this is the case by means of the predicate. The second rule says that if we can say of an inference that it is valid, and we have its premise, we can infer its conclusion. Especially by looking at the first rule, one can easily see how such a predicate could, in principle, enable one to express all valid inferences. The question arose, however, whether such theories could be extended to express the validity of their metainferences.

In particular, in Barrio et al.'s [7], the authors argue that theories based on the logic ST are left with two unwanted choices. Their theories will either underinternalize or overinternalize validity.[26]

For precision, we will define internalization of a metainference as follows:[27]

Definition 3.3. A theory $\mathcal{T}$ *internalizes a metainference* of the form

$$\frac{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$$

If $\mathcal{T}$ proves

$$\Rightarrow (Val(\langle \bigwedge \Gamma_1\rangle, \langle \bigvee \Delta_1\rangle) \wedge \ldots \wedge Val(\langle \bigwedge \Gamma_n\rangle, \langle \bigvee \Delta_n\rangle)) \rightarrow Val(\langle \bigwedge \Gamma\rangle, \langle \bigvee \Delta\rangle).^{28}$$

But then, consider the following negation rule:

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

[26] In a similar vein, Stewart Shapiro [35] charged Priest's LP with overinternalization with the provability predicate. In a theory of arithmetic based on Priest's LP, both $\neg Bew(\langle\vartheta\rangle)$ and $Bew(\langle\vartheta\rangle)$ where ϑ is $\neg Bew(\langle\vartheta\rangle)$, and $Bew(\langle\vartheta\rangle)$ is a shorthand for $\exists x Prf(x, \langle\vartheta\rangle)$. As a result, Priest's system proves that there is a code number that codes the proof of the Gödel sentence ϑ and at the same time, there is no number that codes a proof of the Gödel sentence ϑ. So, even though we can check that a code number is indeed the proof of the sentence ϑ, LP also claims that there is no proof of that exact same sentence. Priest responds by claiming that formal arithmetic must be inconsistent as well. For more details on Priest's response, see ([27], p. 240-243).

Since we are working with a propositional language and have not specified the mechanics of our naming device, let us briefly sketch how it would look in an expanded version of our system. Suppose that we expand our language to include quantifiers and a provability predicate, with a naming device such as that constructed via a Gödel coding method. Suppose also that the provability predicate is partially transparent—enough to produce a provability Liar argument. In such a case, just like the Liar, we would be able to prove $\neg Bew(\langle\vartheta\rangle)$ and $Bew(\langle\vartheta\rangle)$ but only with $\mathbb{A}\vartheta$ tags next to them. Given $\Rightarrow \mathbb{A}$, we can also show that ϑ is paradoxical. So, our system would not overinternalize the provability predicate using the provability Liar and that is without the need to appeal to inconsistent arithmetic.

[27] For more on the internalization of metarules, see [1, 7, 33].

[28] Here, the subscripts are not levels but to indicate multiple (and possibly different) sets of formulas.

Similar to what is shown in [7], the statement expressing the metainference, $Val(\langle \varphi \wedge \bigwedge \Gamma \rangle, \langle \bigvee \Delta \rangle) \to Val(\langle \bigwedge \Gamma \rangle, \langle \bigvee \Delta \vee \neg \varphi \rangle)$ is not provable in the theory of validity built on the non-transitive logic ST. In fact, it is not provable even if the theory is built on a transitive logic (assuming the theory is not trivial). The reason is that VP is too weak to internalize all valid metainferences.

The reason why this occurs is that the rule for introducing the validity predicate on the right is context-free if we follow Beall and Murzi's VP in [9]. Therefore, one could expect to obtain the desired result by strengthening the VP rule (while VD is kept the same as in [9]):

$$\dfrac{Val(\langle \bigwedge \Gamma_1 \rangle, \langle \bigvee \Delta_1 \rangle), \dots, Val(\langle \bigwedge \Gamma_n \rangle, \langle \bigvee \Delta_n \rangle), \Gamma \Rightarrow \Delta}{Val(\langle \bigwedge \Gamma_1 \rangle, \langle \bigvee \Delta_1 \rangle), \dots, Val(\langle \bigwedge \Gamma_n \rangle, \langle \bigvee \Delta_n \rangle) \Rightarrow Val(\langle \bigwedge \Gamma \rangle, \langle \bigvee \Delta \rangle)}\ \text{VP}^{+}$$

$$\dfrac{\varphi \Rightarrow \varphi \qquad \psi \Rightarrow \psi}{Val(\langle \varphi \rangle, \langle \psi \rangle), \varphi \Rightarrow \psi}\ \text{VD}$$

The authors [7] claim that this provides the ST-theorist with so much more than they needed. In fact, such a move enables the following derivation (where the validity Curry, π, is equivalent to $Val(\langle \pi \rangle, \langle \bot \rangle)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\pi \Rightarrow \pi}\ \text{Reflexivity} \qquad \dfrac{}{\bot \Rightarrow \bot}\ \text{Reflexivity}}{Val(\langle \pi \rangle, \langle \bot \rangle), \pi \Rightarrow \bot}\ \text{VD}}{Val(\langle \pi \rangle, \langle \bot \rangle), Val(\langle \pi \rangle, \langle \bot \rangle) \Rightarrow \bot}\ \text{Def of } \pi}{Val(\langle \pi \rangle, \langle \bot \rangle) \Rightarrow \bot}\ \text{Contraction}}{Val(\langle \top \rangle, \langle \pi \rangle), Val(\langle \pi \rangle, \langle \bot \rangle), \top \Rightarrow \bot}\ \text{Weakening}}{Val(\langle \top \rangle, \langle \pi \rangle), Val(\langle \pi \rangle, \langle \bot \rangle) \Rightarrow Val(\langle \top \rangle, \langle \bot \rangle)}\ \text{VP}^{+}}{Val(\langle \top \rangle, \langle \pi \rangle) \wedge Val(\langle \pi \rangle, \langle \bot \rangle) \Rightarrow Val(\langle \top \rangle, \langle \bot \rangle)}\ \wedge\Rightarrow}{\Rightarrow (Val(\langle \top \rangle, \langle \pi \rangle) \wedge Val(\langle \pi \rangle, \langle \bot \rangle)) \to Val(\langle \top \rangle, \langle \bot \rangle)}\ \Rightarrow\to$$

which in a sense states that a validity Curry is cuttable and the empty sequent can be reached in ST (even though this is not true in ST).

This issue of overinternalization, however, is not present for a theory of validity for $\mathcal{CLP}^{\leftmoon}$. The Def of π move would introduce $\mathbb{A}\pi$ and that would stay throughout the whole derivation (assuming that $\mathbb{A}\pi$ is not introduced earlier since VD is a form of transitivity, and assuming we adjust VP+ to allow $\mathbb{A}\pi$ be on the right of the sequent).[29] Thus, in $\mathcal{CLP}^{\leftmoon}$, the conclusion would only state that π is cuttable to get the empty sequent if π is not paradoxical, which accurately represents the situation.[30]

[29] The Weakening move also reintroduces $\mathbb{A}\pi$ as well as $\mathbb{A}\bot$, but $\mathbb{A}\bot$ can be cut and $\mathbb{A}\pi$ can be contracted.

[30] This is not to suggest that there are no other attempts at responding to such a problem. Besides the initial considerations presented in Barrio et al.'s work, other attempts include Ahmad's [1] reformulations of internalization, Hlobil's [19] proposal of a faithful predicate where VP is generalized so as to allow the logic to assume sequents, and Golan's [18] hierarchy of validity predicates. Even though a comparison between these would be interesting, it would take us far afield from our discussion.

4 $\mathcal{CLP}^{\Yin}$ AND $\mathcal{CLP}^{\Yang}$ ARE THE SAME LOGIC

The reader can easily check that all the advantages we showed for $\mathcal{CLP}^{\Yin}$ are also present in $\mathcal{CLP}^{\Yang}$. Similarly, every advantage of $\mathcal{CLP}^{\Yang}$ is an advantage of $\mathcal{CLP}^{\Yin}$. The reason, we contend, is that the two logics are equivalent. In this section, we argue that $\mathcal{CLP}^{\Yin}$ and $\mathcal{CLP}^{\Yang}$ are equivalent, as can be seen in the fact that their provable sequents express the same positions (i.e., the same semantic assumptions).

First compare Reflexivity in $\mathcal{CLP}^{\Yin}$ with $\Rightarrow \triangle$ in $\mathcal{CLP}^{\Yang}$. These two axioms may seem different, but they express the same position. In $\mathcal{CLP}^{\Yang}$, $\varphi \Rightarrow \varphi, \triangle\varphi$ represents the position that for any sentence φ, φ is either true, false, or paradoxical. Without the $\triangle$ on the right, it would not express the third possibility (namely, a sentence can be paradoxical), and thus, it needed to be added. On the other hand, Reflexivity in $\mathcal{CLP}^{\Yin}$ already expresses that third possibility; it expresses the proposition that φ is either true, false, or paradoxical. The reason we do not need $\triangle$ here is that in $\mathcal{CLP}^{\Yin}$, paradoxical is being both true and false, and that is expressed in disjunction since it is an inclusive "or".

Let us turn to Cut. Suppose we want to perform Cut on a sentence, φ, that is not introduced via Weakening.[31] Cut in $\mathcal{CLP}^{\Yin}$ has two premises, one that proves a sentence φ is true and one that proves that it is false with possible context. That entails the possibility that φ is paradoxical, and thus, must be represented as such in the logic. Therefore, in $\mathcal{CLP}^{\Yin}$, it is a requirement that we introduce $\triangle$ to express the possible paradoxicality of such sentences.[32] On the other hand, $\mathcal{CLP}^{\Yang}$ does not need to insert $\triangle$. If φ is not a product of Weakening, as we are assuming, then, because of Lemma 2.2, all sequents already express the possibility of the paradoxicality of φ and thus we do not need to reintroduce it in a Cut move.

As mentioned earlier, the restrictions on left negation, Weakening, and left conditional in $\mathcal{CLP}^{\Yin}$ are there to enforce that unparadoxicality can only be established on sentences that can be proven to be just true or proven to be just false (i.e., sentences equivalent to $\Rightarrow \top$ and $\bot \Rightarrow$), and to ensure that the possibility of a sentence being both true and false does not get written off. Again, such a restriction is not needed in $\mathcal{CLP}^{\Yang}$ because the leftover of $\triangle$ on the right from Reflexivity plays the role of that enforcement; the only way to prove unparadoxicality of sentences in $\mathcal{CLP}^{\Yang}$ is if one can prove the sentences to be just true or prove them to be just false (i.e., sentences equivalent to $\Rightarrow \top$ and $\bot \Rightarrow$), and because of the leftover $\triangle$s, the possibility of sentences being paradoxical are never written off.

These are the only rules that separate $\mathcal{CLP}^{\Yin}$ from $\mathcal{CLP}^{\Yang}$, and we showed how both logics can be interpreted to express the same positions with regards to sequents that result from these rules. The reader can easily check that the meanings of sequents are preserved when the other moves are used.

[31] This is just to avoid instances where Cut is eliminable.

[32] Unless, of course, these sentences cannot be paradoxical (e.g., $\triangle\varphi$), and so, there is no need to introduce $\triangle$ in the conclusion sequent as indicated by the limited Cut rule: $\triangle$Cut.

To drive this point home, suppose that we did not like the current negation, left conditional, and Weakening rules in $\mathcal{CLP}^{\bullet}$. Luckily, there is another method to recapture our regular rules. Simply, replace Reflexivity with:

$$\frac{}{\varphi \Rightarrow \varphi, \varphi \wedge \neg\varphi} \wedge\text{-Reflexivity}$$

This will create a redundancy in the proposition (i.e., φ is either true, false, both, or both). However, it will ensure that negation, left conditional, and Weakening behave normally. Furthermore, Cut can now be fully recaptured since the conclusion sequent will always have a $\triangle$ on the right. Abbreviating $\varphi \wedge \neg\varphi$ as $\triangle\varphi$, the reader can instantly see that it is the same proof system as $\mathcal{CLP}^{\circ}$.

Could there be another motivation for replacing Reflexivity with $\wedge$-Reflexivity in $\mathcal{CLP}^{\bullet}$ other than regaining full generality of the operational rules? We said that one way to read Reflexivity is to read it disjunctively as φ is either true or false. $\mathcal{CLP}^{\circ}$ rejects Reflexivity since φ could be neither true nor false, and replaces it with $\Rightarrow \triangle$ to account for the possibility of the 'neither' option. However, there is another reading for Reflexivity; we can read it as it is not the case that φ is both true and false (i.e., applying a De Morgan to the way we are reading the sequents). For the glut-theorist such a claim is false since there are instances where a sentence is both true and false. So, if we read Reflexivity that way, then the glut-theorist would have to reject Reflexivity and replace it with $\wedge$-Reflexivity to account for the possibility of the "both" option. Of course, rejecting Reflexivity would entail overhauling $\mathcal{CLP}^{\bullet}$'s model theory. In fact, the model theory would have to change to that of $\mathcal{CLP}^{\circ}$ while reading $\triangle\varphi$ as $\varphi \wedge \neg\varphi$. Thus, we would have a reflexive-free but glutty logic.[33] This is all to say that what we presented is the logic $\mathcal{CLP}$ and a theory based on it from two different perspectives. Hence, we join Restall's symmetricalists club [29].[34]

[33] The same reasoning applies the other way; the other reading of sequents and Reflexivity would motivate a non-transitive yet gappy logic.

[34] For other approaches that attempt to bridge the gap between glutty and gappy logics, see [29] and its proof-theoretic semantic approach in terms of assertions, denials, acceptances, and rejections. Restall argues that the advantages and disadvantages are parallel between the glut approaches and the gap approaches. Additionally, in a manuscript [16], Eliana Franceschini shows that there is a translation from ST to K_3, and she makes a similar observation to ours; She notes,

> When an inference, or a metainference, $\Gamma \Rightarrow \Delta$ is valid in a logic **L**, there are two possible interpretations of that fact that produce a significant change in the conceptual role of the metainferences of Cut and Identity: The disjunctive reading [...and] the negated conjunctive reading [...] Identity in the disjunctive (usual) reading expresses an exhaustivity condition on values (standards) [...] It tells us that every valuation either satisfies φ or does not satisfy φ: **an exhaustion constraint** [...] On the other hand, Cut in the disjunctive (usual) reading is an exclusivity condition on values (standards) [...] The disjunctive reading tells us that φ can not be at the same time satisfied and not satisfied. This is an **exclusion constraint**. If we change the reading of the arrow all this is affected. In the negated conjunctive reading, that $\varphi \Rightarrow \varphi$ is valid guarantees that there is no valuation v that both satisfies and not satisfies φ. Now, in this negated-conjunctive reading, Identity is the metainference that expresses an **exclusivity condition** on the values or standards. And, alternatively, on this reading Cut

5 Conclusion and Possible Future Directions

We defined the paradoxicality of a sentence as its inability to consistently receive only the truth value 1 and its inability to consistently receive only the truth value 0. We then showed that there are two ways to represent paradoxicality: One of which takes that to mean that the paradoxical sentence is neither true nor false, and the other one takes that to mean that it is both true and false. For both approaches, we built models that are similar to strong Kleene models but differ ever so slightly.

This resulted in the logic $\mathcal{CLP}$, which was presented along two different perspectives: a gappy reflexive-free $\mathcal{CLP}^{\circ}$ and a glutty transitive-free $\mathcal{CLP}^{\bullet}$. However, the glut-theorist can take the reflexive-free approach and the gap-theorist may choose the non-transitive track. We showed that the theory of truth based on the logic $\mathcal{CLP}$—whichever flavor one chooses—is paradox-free whether we are talking about the Liar, the Curry, revenge paradoxes, or metainferential paradoxes. We also showed that the logic does not overinternalize semantic notions. Overinternalization relies on using 'def of' moves on paradoxical sentences, and such a move always introduces the possibility of paradoxicality of such sentences.

When we introduced the model theory of $\mathcal{CLP}^{\circ}$, we mentioned that there are possible approaches to circumvent the need to use $\triangle$ as a primitive operator. One way is to define a subset of disjunctive sentences that play the role of $\neg\triangle$, call them *bivalent disjunctions*:

Definition 5.1. Any sentence of the form $f(n, \varphi) \vee f(k, \varphi)$ is a *bivalent disjunction* where n is an even number (or 0), k is an odd number, f is a primitive recursive function that takes a number i and a sentence φ and returns the sentence φ with i numbers of negations.

And so, model theoretically, if a disjunction is a bivalent disjunction (i.e., $f(n, \varphi) \vee f(k, \varphi)$), then it is evaluated as 1 when φ is 1 or 0, and evaluated as 0 when φ is n. Thus, now, $\triangle$ is defined as the negation of a bivalent disjunction. Proof-theoretically, the axiom of the sequent calculus becomes:

$$\frac{}{\varphi, f(n, \varphi) \vee f(k, \varphi) \Rightarrow \varphi}\ \vee\text{-Reflexivity}$$

and $\Rightarrow \triangle$ is now derivable from $\vee$-Reflexivity. There must, however, be a restriction on the inverse of the rule $\vee \Rightarrow$ such that $\vee \Rightarrow\uparrow$ is restricted to non-bivalent disjunctions, for otherwise, the proof-system would collapse to LK.

All of the results we have shown for $\mathcal{CLP}^{\circ}$ are mirrored in this approach. One advantage of such an approach, nevertheless, is that the language stays

> tells us that so long as we have a model of $\Gamma \Rightarrow \Delta$, φ has to be able either to be satisfied or not satisfied. Cut, on the negated-conjunctive reading, is an **exhaustivity condition** on values or standards. ([16], pp. 14-15)

Similarly, Ryan Simonelli [36] argues that the left-sided fragment of bilateral K_3 is just a notational variant of ST, and that in a bilateral setting, rejecting Cut amounts to rejecting a bilateral principle of excluded middle rather than transitivity.

the same as that of standard propositional language and we can define the Par predicate in terms of the Tr predicate. The downside is that truth-functionality is lost (though compositionality is not lost). This is because, now, in order to determine the truth value of a disjunction, we must also know the syntactic structure of the disjuncts on top of knowing the truth value of the disjuncts.

There is, however, another way to retain truth-functionality without losing interdefinability of $\triangle$ with the connectives. Because both the value n and 0 are undesignated, we can 'collapse' the middle values in the tables.[35] For example, the disjunction in $\mathcal{CLP}^{\wp}$ would be interpreted as:

$\varphi \vee \psi$	1	n	0
1	1	1	1
n	1	0	n
0	1	n	0

This way, a bivalent disjunction is not treated as a special sentence but can be read off—diagonally—of the same table.

Such an approach retains truth-functionality, but also recovers more instances of Reflexivity. In this approach, not all sequents need $\triangle$ appearing in them; for example, $\varphi \vee \varphi \Rightarrow \varphi$ would become valid. Notice, however, that the fact that φ could be paradoxical is now implicit in the antecedent $\varphi \vee \varphi$. So, when we read the sequent as either $\varphi \vee \varphi$ is false or φ is true, it is implicit in the falsity of $\varphi \vee \varphi$ that φ can either be false or paradoxical, given that $n \vee n$ and $0 \vee 0$ both give you a false disjunction. Another approach is to also collapse the other n's appearing in the previous table. Whether they collapse to 1 or to 0, it would require further justification on why they should collapse, and this could validate more sequents.

If such an approach is taken, then our understanding of paradoxicality must be adjusted. Paradoxicality would only apply to atomics and negation(s) of atomics, and so, we would have to say that κ is paradoxical (where κ is a Curry) while $Tr(\langle\kappa\rangle) \to \bot$ is not. Whether this is an advantage over our current approach is yet to be explored. Nevertheless, such an approach would create an asymmetry between inferences of the first level and higher level inferences; in the first inferential level, some formulas might be paradoxical and receive the value n. However, since we take sequents to be metaconditionals and are defined the same way as conditionals, there can be no paradoxical sequents in metainferences.

Interestingly, if we collapse the 'b's to '1's in $\mathcal{CLP}^{\bullet}$ (even if it is done only on the conditional), then the logic would get even closer to the logic ST while avoiding its problems (i.e., revenge paradoxes, metainferential paradoxes, and overinternalization of validity). However, it would enable us to derive the liar and its negation by themselves, and hence, we would no longer be able to read '$\Rightarrow \varphi$' as 'φ is true only'.

Therefore, one possible future exploration is to assess the merits and disadvantages of each approach. What is clear at the moment is that such collapse

[35] For more on collapsing the middle value (i.e., n or b) to a boolean value, see [13].

methods would require further philosophical justification. Other future projects might include showing how to expand the language to a first-order language and how the logic $\mathcal{CLP}$ can handle soritical paradoxes as well as establishing their paradoxicality. Moreover, we want to explore the relation between $\mathcal{CLP}$ and limitative theorems such as Gödel's incompleteness theorems (i.e., develop footnote 26 in detail). Finally, our starting point in this logic was that of sequents. We expect the approach can be expanded to include a lower level to include a K_3-like level (see, [14, 15, 22]) in the $\mathcal{CLP}^{\text{◐}}$ version and an LP-like level (see, [3, 26, 27]) in the $\mathcal{CLP}^{\text{◑}}$ version. If $\mathcal{CLP}$ is expandable to include the K_3-like and LP-like versions, a further question concerns the place of FDE in the picture presented.[36]

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Appendix: Completeness

Here is the proof of completeness in Theorem 2.1:

Proof. The proof mostly follows [31].

We will extend trees upward as follows:

- If the sequent is $\Gamma_n \Rightarrow_n \neg\varphi_n, \Delta_n$, then it is extended to

  $\Gamma_n, \varphi_n \Rightarrow_n \neg\varphi_n, \Delta_n$

- If the sequent is $\Gamma_n, \neg\varphi_n \Rightarrow_n \Delta_n$, then it is extended to $\Gamma_n, \neg\varphi_n \Rightarrow_n \varphi_n, \Delta_n$
- If the sequent is $\Gamma_n, \varphi_n \vee \psi_n \Rightarrow_n \Delta_n$, then it is extended to $\Gamma_n, \varphi_n \vee \psi_n, \varphi_n \Rightarrow_n \Delta_n$ and $\Gamma_n, \varphi_n \vee \psi_n, \psi_n \Rightarrow_n \Delta_n$
- If the sequent is $\Gamma_n \Rightarrow_n \varphi_n \vee \psi_n, \Delta_n$, then it is extended to $\Gamma_n \Rightarrow_n \varphi_n, \psi_n, \varphi_n \vee \psi_n, \Delta_n$
- If the sequent is $\Gamma_n, \triangle\varphi_n \Rightarrow_n \Delta_n$, then it is extended to $\Gamma_n, \varphi_n, \triangle\varphi_n \Rightarrow_n \varphi_n, \Delta_n$
- If the sequent is $\Gamma_n \Rightarrow_n \Delta_n, \triangle\varphi_n$, then it is extended to $\Gamma_n \Rightarrow_n \Delta_n, \triangle\varphi_n, \varphi_n$ and $\Gamma_n, \varphi_n \Rightarrow_n \Delta_n, \triangle\varphi_n$
- If the sequent is $\Gamma_n, Tr(\langle\varphi_n\rangle) \Rightarrow_n \Delta_n$, then it is extended to $\Gamma_n, Tr(\langle\varphi_n\rangle), \varphi_n \Rightarrow_n \Delta_n$, and if the sequent is $\Gamma_n \Rightarrow_n \Delta_n, Tr(\langle\varphi_n\rangle)$, then it is extended to $\Gamma_n \Rightarrow_n \Delta_n, Tr(\langle\varphi_n\rangle), \varphi_n$
- If the sequent is $\Gamma_n, Par(\langle\varphi_n\rangle) \Rightarrow_n \Delta_n$, then it is extended to $\Gamma_n, Par(\langle\varphi_n\rangle), \triangle\varphi_n \Rightarrow_n \Delta_n$, and if the sequent is $\Gamma_n \Rightarrow_n \Delta_n, Par(\langle\varphi_n\rangle)$, then it is extended to $\Gamma_n \Rightarrow_n \Delta_n, Par(\langle\varphi_n\rangle), \triangle\varphi_n$

To simplify the proof, we highlight the fact that for any starting axiom $\Rightarrow \triangle$, there are finitely many possible inverse-only rules that can be applied to it. Call the n-many inverse rules steps applied to $\Rightarrow \triangle$, *Descendants of* $\Rightarrow \triangle$.

Suppose that $\nvdash_{t\mathcal{CLP}^{\wp}} \Gamma_n \Rightarrow_n \Delta_n$.

If the tree is closed (reaches a weakened version of our axiom $\Rightarrow \triangle$ or a weakened version of a descendant of $\Rightarrow \triangle$), then the tree is a derivation in $t\mathcal{CLP}^{\wp}$. Thus, we reach a contradiction.

If the tree is not closed, then there will be counterexamples. Notice that the process of extending sequents described above can be followed ω-times. Thus, take the union of everything occurring on the left of any sequent in the open branch and call it Γ^{ω}, and similarly for the right to obtain Δ^{ω}. Let the resulting sequent be $\Gamma^{\omega}_n \Rightarrow_n \Delta^{\omega}_n$. Such a sequent is used to define the model. (Keep in mind that a $t\mathcal{CLP}^{\wp}$ counterexample is when no sentence receives 0 in the antecedent of the sequent and no sentence receives 1 in the consequent of the sequent).

Case 1: there are no $\triangle$ sentences whatsoever

- Simply let every sentence receive the value n.

Case 2: A $\triangle\varphi_n$ appears in Γ^{ω}_n but not in Δ^{ω}_n

- If φ_n is a $\triangle$ sentence or negation(s) of a $\triangle$ sentence, then that sequent is provable *contradiction*.

- Otherwise, let φ_n and every sentence receive the value n. This way, there is no 0 in Γ_n^ω and no 1 in Δ_n^ω.

Case 3: A $\mathbb{A}\varphi_n$ appears in Δ_n^ω but not in Γ_n^ω

- If φ_n appears in Γ_n^ω and in Δ_n^ω, then it is a form of weakened $\Rightarrow \mathbb{A}$, and thus, derivable, *contradiction*.
- If φ_n appears in Γ_n^ω only, then let it take the value 1, and everything else the value n.
- If φ_n appears in Δ_n^ω only, then let it take the value 0, and everything else the value n.
- If φ_n is a $\mathbb{A}$ sentence or negation(s) of a $\mathbb{A}$ sentence, then $\mathbb{A}\varphi_n$ will receive the value 0 no matter what.

Case 4: A $\mathbb{A}\varphi_n$ appears in both Γ_n^ω and Δ_n^ω

- It is a derivable sequent, *contradiction*.

As for how to ensure that the truth predicate behaves as expected, we follow the "fixing" method as found in Ripley's [31]. This procedure guarantees that there is a countermodel to $\Gamma_n^\omega \Rightarrow_n \Delta_n^\omega$. Thus, there is a $t\mathcal{CLP}^{\wp}$-model such that $\not\models_{t\mathcal{CLP}^{\wp}} \Gamma_n \Rightarrow_n \Delta_n$. □

One thing to note is that in this previous proof we are assuming that $t\mathcal{CLP}^{\wp}$ contains the rules of $\mathcal{CLP}^{\wp}$ and truth and paradoxicality rules, but they do not contain naming devices that establish Id Est (Def of) moves. As a result, we did not have to worry about Cut since it is taken care off by the tree extension rule of $\Gamma_n \Rightarrow_n \Delta_n, \mathbb{A}\varphi_n$. Nevertheless, if one is insistent in having a completeness proof that includes Id Est moves, then there is still a way. Unlike classical settings, instances of Cut always leave some traces behind them in our system (unless the cut formulas were weakened in, and so Cut here would be eliminable), enabling us to provide rules for extending the trees (like the tree rule for $\Gamma_n \Rightarrow_n \Delta_n, \mathbb{A}\varphi_n$ above). And so, we only need to add the following tree-extension rules to the previous instructions:

- If the sequent is $\Gamma_n \Rightarrow_n \rho_n, \Delta_n$ where ρ_n is a "diagonal sentence" by stipulation (see [30]'s τ naming function), then extend the tree to $\Gamma_n, \pi_n \Rightarrow_n \rho_n, \Delta_n$ and to $\Gamma_n \Rightarrow_n \pi_n, \rho_n, \Delta_n$ where π_n is ρ_n's equivalence by stipulation (for example, if ρ_n is λ, then π_n is $\neg Tr(\lambda)$). If any of these two nodes are weakened Id est of ρ_n (i.e., $\mathbb{A}\rho_n$ must be in Δ_n for $\Gamma_n, \pi_n \Rightarrow_n \rho_n, \Delta_n$ to be a weakened Id est, then the branch of that node halts.
- If the sequent is $\Gamma_n, \rho_n \Rightarrow_n \Delta_n$ where ρ_n is a "diagonal sentence", then extend the tree into the two branches $\Gamma_n, \rho_n \Rightarrow_n \pi_n, \Delta_n$ and $\Gamma_n, \pi_n, \rho_n \Rightarrow_n \Delta_n$ where π_n is ρ_n's equivalence by stipulation. Halt a branch if the branch is a weakened Id Est.

To reiterate, other Cut moves are handled by the tree extension of $\Gamma_n \Rightarrow_n \Delta_n, \mathbb{A}\varphi_n$. No further extension rules are needed.